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Multilattice Tilings and Coverings

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ABSTRACT

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Let L be a discrete subgroup of $\mathbb{R}^n$ under addition. Let D be a finite set of points including the origin. These two sets will define a multilattice of $\mathbb{R}^n$. We explore how to generate a periodic covering of the space $\mathbb{R}^n$ based on L and D. Additionally, we explore the problem of covering when we restrict ourselves to covering $\mathbb{R}^n$ using only dilations of the right regular simplex in our covering. We show that using a set $D = \{\bar{0}, d\}$ to define our multilattice the minimum covering density is $5 - \sqrt{13}$. Furthermore, we show that when we allow for an arbitrary number of displacements, we may get arbitrarily close to a covering density of 1.

Keywords: multilattice, covering, density, word-length, simplex

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Chapter 1. The Initial Covering Problem

The problem of how to cover a space or how to pack given objects into a space is a problem that mathematicians have been interested in for quite some time [1]. This naturally arises from the cannonball packing problem. Given a set of cannonballs, what is the most effective way to pack them into a given space? This first arose when the question of how to store the greatest amount of cannonballs on ships was explored. The Kepler conjecture states that spheres centered on the face-centered cubic lattice result in the best possible packing density in $\mathbb{E}^3$. The Kepler conjecture was later proved by Thomas Hales and Samuel Ferguson [1].

Exploration of this topic naturally led to the similar question of how to cover $\mathbb{E}^3$ with overlapping spheres. In other words, if we take a set of spheres that all share the same radius, how should we place them most efficiently so that every point of $\mathbb{E}^3$ is contained in at least one sphere? In an effort to answer this question, mathematicians began to explore different methods of packing and covering with a variety of shapes and arrangements of them. Clearly, there are many different ways to cover a given space, but which is most efficient? We will focus on one method that has been used to obtain a covering with simplices, and several results related to this problem.

1.1 Packing and covering

To approach the problem of efficiently covering spaces by using lattices, we first establish some key definitions and results from previous approaches to the topic. While we will work on covering the space $\mathbb{E}^n$, in order to make some results easier to prove, we will use $\mathbb{R}^n$ with the dot product as the inner product. Note that $\mathbb{R}^n$ has the same underlying set as $\mathbb{E}^n$, but allows us to define an origin. Having an origin will be important for calculations later. We also define a convex body.

Definition 1.1. A *convex body* is a compact set C of points in $\mathbb{R}^n$ such that for any two points x and y in C, the line segment that joins them is also contained in C.

We will use convex bodies and their translates to cover our space. One method of using a given convex body to obtain a cover for a given space is to place translates of the convex body in periodic patterns that result in every point of the space being contained in at least one of the translates. One method that many are familiar with is a periodic pattern that results in a lattice.

Definition 1.2. A *Lattice* L of rank i ($i \leq n$) is a discrete subgroup of $\mathbb{R}^n$, under addition, generated by all integer combinations of $\mathbb{R}$-linearly-independent vectors $\{v_1, v_2, \ldots, v_i\}$. The lattice L is said to be *full rank* if $i = n$.

One example of a lattice is $\mathbb{Z}^n$. The lattice $\mathbb{Z}^n$ is generated by the vectors $\{e_1, e_2, \ldots, e_n\}$. Notice that this is a full rank lattice in $\mathbb{R}^n$ as n vectors are required to generate $\mathbb{Z}^n$. Another lattice is the set of points $L = \{(a, b, b) \mid a, b \in \mathbb{Z}\}$. This lattice will be generated by the vectors $(1, 0, 0)$ and $(0, 1, 1)$. This is not a full rank lattice, as it is generated by only 2 vectors, but is a subgroup of $\mathbb{R}^3$. Another example, perhaps the most famous, is the *face-centered cubic lattice*. The face-centered cubic lattice is generated by the three vectors $(-1, -1, 0)$, $(1, -1, 0)$, and $(0, 1, -1)$ and is a full rank lattice in $\mathbb{R}^3$. This lattice is seen in chemistry and physics in the formation of crystals. It is also the lattice that gives rise to the highest possible packing density of cannonballs in the Kepler conjecture. Now that we have defined a lattice, we approach the problem of covering the space $\mathbb{R}^n$ by using a given lattice to define translates of a single convex body. Let us place a convex body C in $\mathbb{R}^n$ such that the origin is contained in C. Let us now take all translates of C by each element of our lattice. It is possible, given the proper choice of convex body, that this periodic pattern of simplices will contain every point of $\mathbb{R}^n$. This will result in what is known as a lattice covering.

Definition 1.3. A *lattice covering* of $\mathbb{R}^n$ is a covering obtained by placing a convex body S in $\mathbb{R}^n$ such that the origin is contained in S, and the union of all translates of S by elements of a full rank lattice L is $\mathbb{R}^n$. We will often write $L + S = \mathbb{R}^n$ to express that S and L generate a cover of $\mathbb{R}^n$.

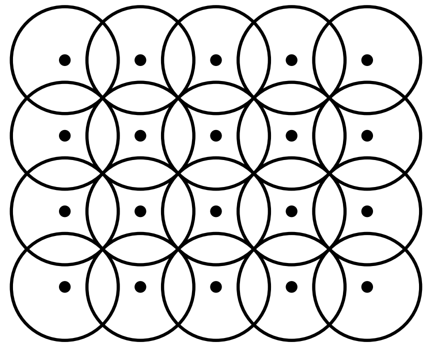

Figure 1.1: Covering $\mathbb{R}^2$ with disks

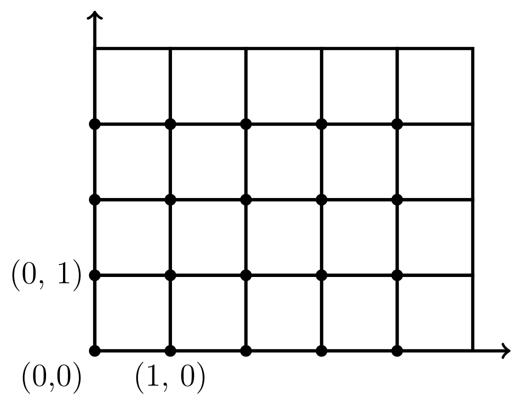

Figure 1.2: Covering $\mathbb{R}^2$ with half-open squares

For example, in $\mathbb{R}^2$ let L be the full-rank lattice $\mathbb{Z}^2$ and the convex body S to be the disk of radius $\frac{1}{\sqrt{2}}$ centered at the origin. Now we take all translates of S by any element of $\mathbb{Z}^2$, and we get the set of all disks of radius $\frac{1}{\sqrt{2}}$ centered at integral points. This will result in a covering of the space $\mathbb{R}^2$, pictured in Figure 1.1, and so is a lattice covering. Some points in $\mathbb{R}^2$ are contained in multiple spheres in our covering.

Let us look at another covering defined using a lattice. Suppose that instead of using disks, we had taken the set of points $[0, 1) \times [0, 1)$. This is a square, but with two adjacent edges not included. We will call this the *half-open square*. We place half-open squares of side length 1 at each lattice point, such that the vertices of each square are lattice points. This will cover the space $\mathbb{R}^2$. This covering is illustrated in Figure 1.2. This covering results in each point belonging to exactly one of our half-open squares. As the set used to cover is not compact, this is not a lattice covering. We may obtain a lattice covering by taking the

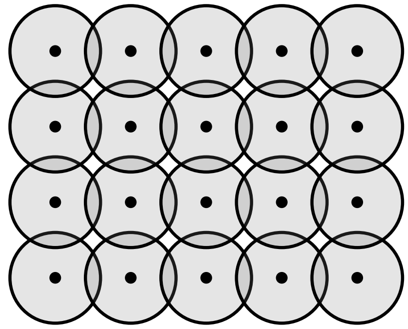

Figure 1.3: Failing to cover $\mathbb{R}^2$ with disks

closure of the half-open square and placing the vertices in the same manner.

Now we look at an example of a non-covering. Suppose that we once again take the lattice $\mathbb{Z}^2$. Let S be the disk of radius $\frac{3}{5}$ centered at the origin. We take all translates of S by the elements of L. However, note that we now have regions of $\mathbb{R}^2$ that are not covered by any of the translates of S. This non-covering is illustrated in Figure 1.3.

We are interested in coverings for our space. So we define a method that will check if a given lattice and convex body generate a lattice covering. To do this, we first define a tile.

Definition 1.4. Let S be a set. Let $P = \{S_1, S_2, \ldots\}$ be a partition of the set S. A *transversal* is a set T such that for every $S_i \in P$, there is exactly one $x_i \in T$ such that $x_i \in S_i$.

Definition 1.5. A *Tile* is a transversal T of the cosets of the group $\mathbb{R}^n/L$, for a full-rank lattice L.

For any $x \in \mathbb{R}^n$, x may be written uniquely as $x' + g$ where $x' \in T$ and $g \in L$ for a tile T of our lattice L. We need the lattice to be full rank, otherwise $\mathbb{R}^n/L$ will not have finite volume. We can think of finding a tile as taking a set of points T in $\mathbb{R}^n$ such that the restriction of the quotient map to T will be a bijective map between T and $\mathbb{R}^n/L$. We now look at an example of a tile. Let $n = 1$ and let $L = \mathbb{Z}$. This is a discrete subgroup of $\mathbb{R}$ and is clearly rank 1. Now let us take the half open interval $[0, 1)$. Notice that this is a transversal of the cosets of $\mathbb{R}/\mathbb{Z}$, or in our case $\mathbb{R}^n/L$. Thus, this half open interval is a tile of the lattice $\mathbb{Z}$. We note that if we place the interval $[i, i + 1)$ at each of the points in $\mathbb{Z}$, we will certainly cover $\mathbb{R}$ as every real number is an element of an interval of this type. We also note that there is no overlap between any of the intervals of this type. Let us consider one more example of a tile before moving on. We take the space $\mathbb{R}^2$. Take the lattice L generated by $v_1 = (1,0)$ and $v_2 = (0,2)$. Now we look at the quotient group $\mathbb{R}^2/L$. For any point $(a,b) \in \mathbb{R}^2$, the point $(a - m_1, b - 2m_2)$, with $m_1, m_2 \in \mathbb{Z}$, maps to the same coset in the quotient group. So we may choose a representative (x,y) for the coset $(a,b) + L$ such that $x \in [0,1)$ and $y \in [0,2)$. So the set $[0,1) \times [0,2)$ is a transversal of the cosets of $\mathbb{R}^2/L$ and thus a tile.

We note that a tile of a lattice L will always give us a covering of $\mathbb{R}^n$. This leads us to the following.

Theorem 1.6. *Any convex body C generates a lattice covering of $\mathbb{R}^n$ with the lattice L if and only if C contains at least one tile of the lattice L.*

Proof. For the forward direction, let C be a convex body that covers with the lattice L. We now show that C must contain a tile. Let x be any point of $\mathbb{R}^n$. Then x is contained in C translated by some $g \in L$, as $L + C$ is a lattice covering. In other words, $x = c + g$ for some $c \in C$ and $g \in L$, or $x - g = c$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^n/L$ be the standard quotient map. Then $\pi(x) = \pi(x) - \pi(0) = \pi(x) - \pi(g) = \pi(x - g) = \pi(c)$. However, $x - g = c$ lies inside the convex body C. We then have that $\pi(C) = \mathbb{R}^n/L$. Now let T be a tile of the lattice L. We note that for any point $x' \in T$, there exists a point $y' \in C$ such that $y' = x' + g$ for some g in our lattice. Let T' be the set of all y'. Clearly $T' \subseteq C$. We also note that $\pi(x') = \pi(y')$. So T' is also a transversal of $\mathbb{R}^n/L$, and so is a tile. C contains a tile.

For the other direction, let T be a tile of the lattice L. Let C be a convex body that contains T. We now show that $C + L$ will give us a lattice covering. Choose any point $x \in \mathbb{R}^n$. We note that x must be contained inside of a translate of our tile T by a lattice element g. Call this translation $T + g$. We note that $T \subseteq C$. So we also have that $T + g \subseteq C + g$, where $C + g$ is the translation of C by the element g. So x is also contained in $C + g$. This is true for every x, so $C + L$ must be a covering of $\mathbb{R}^n$. $\square$

It is important to note that a tile for a given lattice is not unique, it is just a transversal of the cosets $\mathbb{R}^n/L$. So it is possible that we may have a convex body which does not contain a given tile of L, but still gives a covering when paired with L.

As an example we again work in $\mathbb{R}$ with $\mathbb{Z}$ as our lattice. Suppose we had taken the convex body $[-\frac{1}{2}, \frac{1}{2}]$. This does not contain the tile $[0, 1)$, but it does contain the tile $[-\frac{1}{2}, \frac{1}{2})$. So long as a single tile of L is found in the chosen convex body, we have a cover of $\mathbb{R}^n$. The convex body $[-\frac{1}{2}, \frac{1}{2}]$ will still provide a cover with the lattice $\mathbb{Z}$.

We also note that tiles need not be connected. We may choose the tile $(a - \frac{1}{4}, a + \frac{1}{4}] \cup (a + \frac{5}{4}, a + \frac{7}{4}]$. This is a tile for $\mathbb{Z}$, as it is a transversal of each of the cosets of $\mathbb{R}/\mathbb{Z}$. It is not a connected set, but is a union of connected sets, which contains a tile (itself). Any convex body which contains this tile will also give us a lattice covering for $\mathbb{R}/\mathbb{Z}$. We see that since a lattice does not have a unique tile, it is possible to choose many different convex bodies to cover our space. With this in mind we turn to one way to obtain a tile, and the approach we will use moving forward.

1.2 The subtraction Construction

In order to obtain a tile for a given lattice L, we turn to a previously developed method called the *subtraction construction*[2]. To begin the subtraction construction, we denote by F all points of $\mathbb{R}^n$ with each coordinate non-negative (the closed first orthant). (As we work through the process, we will perform the subtraction construction to obtain a tile of the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$). Let a lattice L be given. Take all the points of L whose coordinates have a positive sum and call this set L^+. (In $\mathbb{Z}^2$, this is the set $\{(x, y) \in \mathbb{Z}^2 \mid x+y > 0\}$). If x_i denotes the ith coordinate of the point x, $\sum_{i=1}^{n} x_i > 0$ for each $x \in L^+$. For any $x \in L^+$, it must be true that $-x \notin L^+$. Now let $L^0 = \{x \in L \mid \sum_{i=1}^{n} x_i = 0, x \succ -x$ under the lexicographical order$\}$. (In $\mathbb{Z}^2$, $L^0 = \{(x, -x) \in \mathbb{R}^2 \mid x \in \mathbb{Z}^-\}$). Let $L^{0+} = L^0 \cup L^+$. ($L^{0+} = \{(x, y) \in \mathbb{Z}^2 \mid x + y > 0\} \cup \{(x, -x) \mid x > 0\}$). Let $L^{0+} + F = \{x' + y' \in \mathbb{R}^n \mid x' \in L^{0+}, y' \in F\}$. In other words, $L^{0+} + F$ is the set of points that are contained in the first orthant translated by some element of L^{0+}. ($L^{0+} + F = \{(x+a, y+b) \in \mathbb{R}^2 \mid (x,y) \in L^{0+}, a \in \mathbb{R}^{0+}, b \in \mathbb{R}^{0+}\}$). Let $T = F \setminus (L^{0+} + F)$. (For $\mathbb{Z}^2$, $(0,1), (1,0) \in L^{0+}$, and $T = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x, y < 1\}$. Then T is a tile of the space $\mathbb{R}^n$ with the lattice L [2]. T is said to be a tile obtained from the subtraction construction. In order to discuss the significance of this tile, we need to define Manhattan diameter.

Definition 1.7. The *Manhattan diameter* between x and y in $\mathbb{R}^n$ is $\lambda = \sum_{i=1}^{n} |x_i - y_i|$ where $x_i - y_i$ is the difference in the ith coordinate of x and y. The Manhattan diameter of a bounded subset C of $\mathbb{R}^n$ is the smallest λ such that C is a subset of a right regular simplex defined by the points $\{x, x + \lambda e_1, \ldots, x + \lambda e_n\}$ for some $x \in \mathbb{R}^n$. We will write this as $\mathrm{Mdiam}\,(T)$

We note that if S is the right regular simplex defined by the points $\{x, x+\lambda e_1, \ldots, x+\lambda e_n\}$, then any point x' contained in S will have Manhattan diameter at most λ from the vertex x. So the Manhattan diameter of a tile will give us the size of the right regular simplex required to contain the tile.

Theorem 1.8 (Forcade & Lamoreaux)**.** *A tile obtained from the subtraction construction will have a minimal Manhattan diameter for any tile type of the lattice L.*

Specifically, if λ is the Manhattan diameter of the tile, then the tile is contained in the right regular simplex of area $\frac{\lambda^n}{n!}$ defined by the points $\{0, \lambda e_1, \ldots, \lambda e_n\}$. So the Manhattan diameter of the tile defines the minimum volume of a right regular simplex S that will cover $\mathbb{R}^n$ with L. So if S' is a right regular simplex with smaller volume than S, then $S' + L \neq \mathbb{R}^n$. For the proof of this theorem, we refer the reader to [2]. We will generalize this theorem to the setting of multilattices (Theorem 2.7).

We will now use the subtraction construction to find a tile of a few given lattices. First, let us take the lattice $\mathbb{Z}$. We have seen several examples of tiles for this lattice. To begin we note that since this is a one dimensional lattice, L^{0+} will be the set of all lattice points that are non-negative numbers, or $L^{0+} = \mathbb{N} \cup \{0\}$. We note that F will be the set $[0, \infty)$.

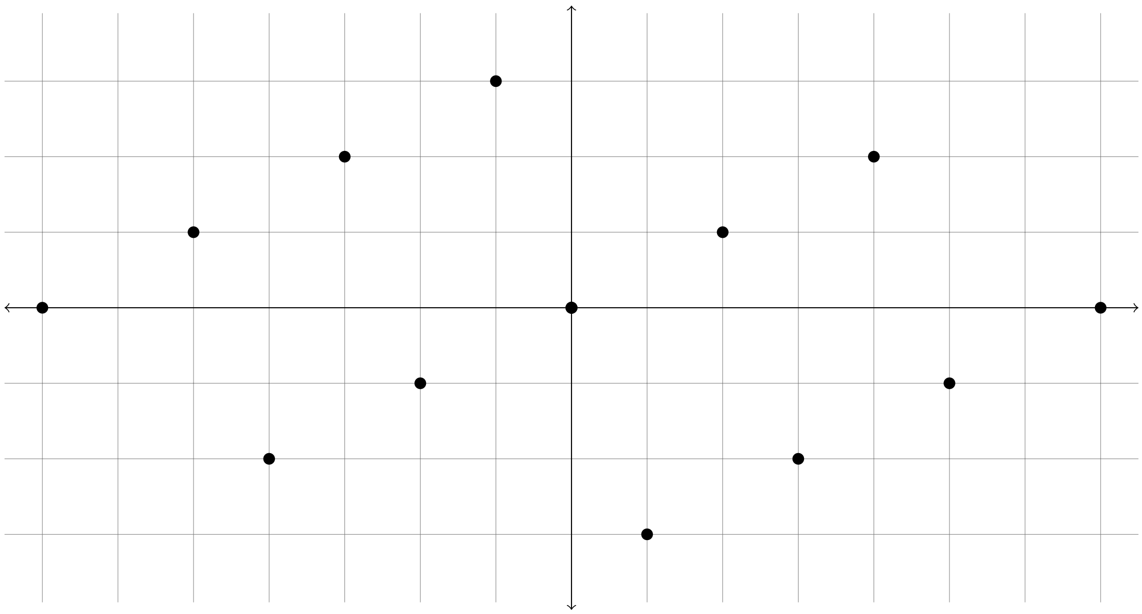

Figure 1.4: The lattice L generated by $(2,1)$ and $(-1,3)$

So our set $L^{0+}+F$ will be the translations of $[0,\infty)$ by a positive integer. Since L^{0+} is well-ordered, it has a least element, namely 1. So $L^{0+}+F=[1,\infty)$. Our tile becomes $[0,\infty)\setminus[1,\infty)=[0,1)$. It should be noted that our tile obtained this way is meant to define a minimal right regular simplex to cover our space. So this tile will allow us to choose a 1-simplex that will cover $\mathbb{R}$ with minimum density. The simplex will be $[0,1]$.

We now look at a more interesting example. Let us take the lattice generated by $v_1=(2,1)$ and $v_2=(-1,3)$, pictured in Figure 1.4. This is a rank 2 lattice, so we start with F as the first quadrant, which is colored red in Figure 1.5. We also note that the hyperplane defined by $\sum_{i=1}^{n} x_i=0$ will be the line $y=-x$, also pictured in Figure 1.5. We note that L^{0+} be the set $\{m_1v_1+m_2v_2 \mid 3m_1+2m_2>0, m_1,m_2\in\mathbb{Z}\}$. Notice that the set $L^{0+}+F$ will be the union $\bigcup_{x\in L^{0+}} x+F$, shown in Figure 1.6 as the gray region. The red region in Figure 1.6 is the tile for this lattice from the subtraction construction. In other words, the red region in Figure 1.6 is $F\setminus(L^{0+}+F)$. Notice that while there are infinitely many points in L^{0+}, there are only finitely many of them we need to define our tile T in this way. In this case they are the points $(2,1),(-1,3),(3,-2)$. These points are often called *blockers*. Informally, a blocker is a point in L^{0+} that "blocks" our tile from increasing in a

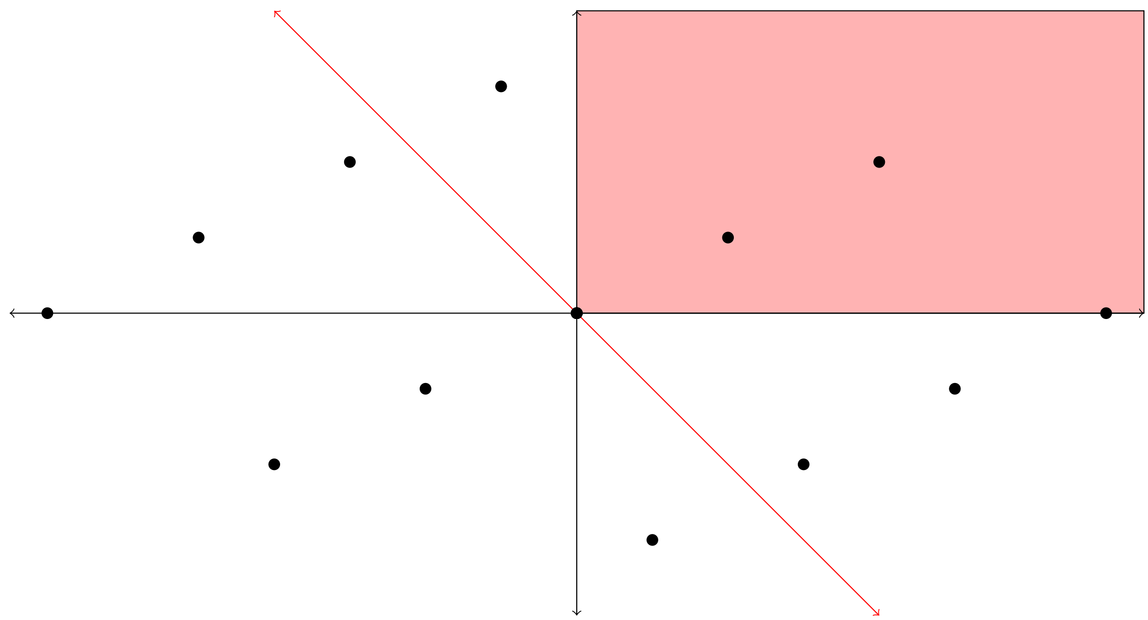

Figure 1.5: The set F in $\mathbb{R}^2$ and the hyperplane $y = -x$

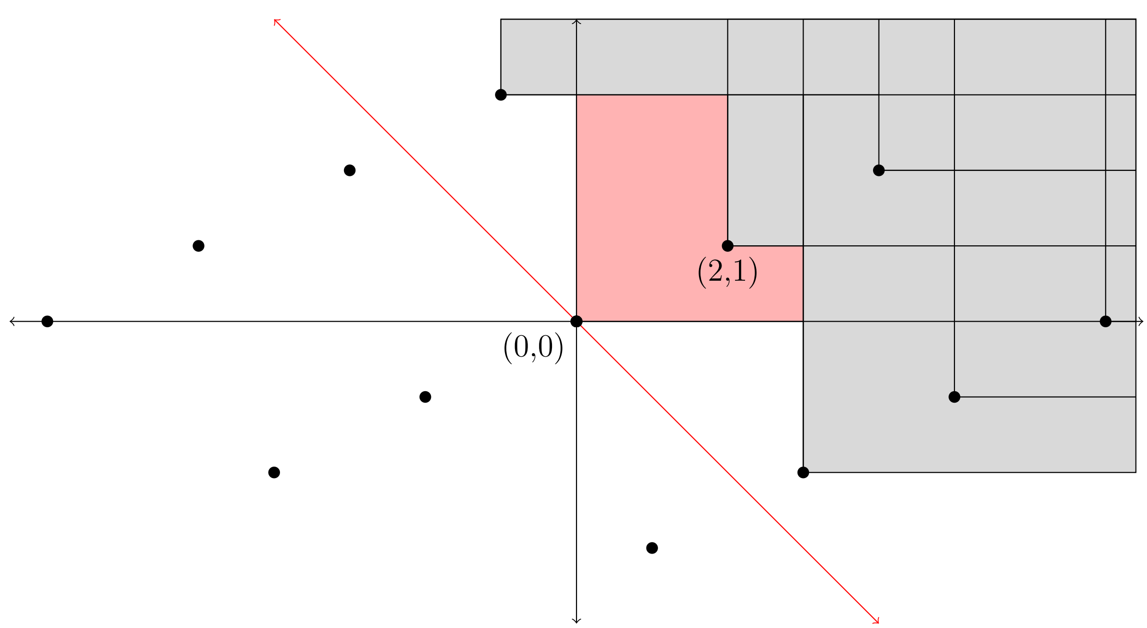

Figure 1.6: The set $L^{0+} + F$ and tile T for the lattice generated by $(1, 2)$ and $(-1, 3)$

given direction. For example, here the blocker $(-1,3)$ for our tile here "blocks" our tile from having points with a y-coordinate greater than 3. The blocker $(2,1)$ "blocks" our tile from having points with an x-coordinate greater than 2 and a y-coordinate greater than 1. In this way we get one corner of our tile as $(2,3)$. A blocker that has strictly positive coordinates is called the *squinch*.

1.3 Density

With the subtraction construction, we have a way of finding a tile for a given lattice. Now that we may check if a convex body will give us a lattice covering, we want to find which convex body will cover with the least amount of overlap. In other words, we want to find a covering that minimizes the volume of the set of points in a tile T that are covered by more than one convex body in our lattice covering. To approach this problem, we will also define a few terms.

Definition 1.9. Let the lattice L be generated by $\{v_1, \ldots, v_n\}$, and L be a full rank lattice in $\mathbb{R}^n$. The *covolume of a lattice* is the absolute value of the determinant of the matrix of the form $\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ where v_i are column vectors.

Theorem 1.10. *The volume of any connected and bounded tile of a lattice L is equal to the covolume of the lattice L.*

Proof. We let T be the parallelepiped $[0,1)v_1 \times \cdots \times [0,1)v_n$. Then T will have volume $\det \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$. We now show that T is a tile. First we show that under the standard projection map $\pi : \mathbb{R}^n \to \mathbb{R}^n / \langle v_1, v_2, \ldots, v_n \rangle$, $\pi \mid_T$ is injective. Choose $x, y \in T$ such that $x \neq y$. Now we show that $\pi(x) \neq \pi(y)$. Assume by way of contradiction that $\pi(x) = \pi(y)$. This means that $x = y + g$, where g is some elements of the group $\langle v_1, v_2, \ldots, v_n \rangle$. Note that $\langle v_1, v_2, \ldots, v_n \rangle = L$, so g is a lattice point. Since L is a full-rank lattice, we have that $\{v_1, \ldots, v_n\}$ is also a basis for $\mathbb{R}^n$. Since these vectors form a linearly independent basis for $\mathbb{R}^n$, we may uniquely express x and y in terms of this basis. Let $x = r_1 v_1 + r_2 v_2 + \cdots + r_n v_n$

and let $y = s_1v_1 + s_2v_2 + \cdots + s_nv_n$, where $r_i, s_i \in \mathbb{R}$. Then $r_i \neq s_i$ for some i. For any $g \in L$, we may write g as $m_1v_1 + m_2v_2 + \cdots + m_nv_n$, with $m_i \in \mathbb{Z}$. Since $x = y + g$ for some g, we have

$$r_1v_1 + \cdots + r_nv_n = s_1v_1 + \cdots + s_nv_n + m_1v_1 + \cdots + m_nv_n.$$

The basis for L is linearly independent, so $r_iv_i = (s_i + m_i)v_i$ for each i. Since $x, y \in T$, $r_i \in [0, 1)$ and $s_i \in [0, 1)$. Thus it must be the case that $m_i = 0$ for all i. But this gives us that $s_i = r_i$, a contradiction. So π must be injective when restricted to T.

We now show that $\pi : T \to \mathbb{R}^n/L$ is surjective. The quotient map $\pi : \mathbb{R}^n \to \mathbb{R}^n/L$ is surjective. Choose any $\bar{x}$ in $\mathbb{R}^n/L$. There must be some x in $\mathbb{R}^n$ such that $\pi(x) = \bar{x}$. Since $\{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$, we may express x as $\alpha_1v_1 + \alpha_2v_2 + \cdots + \alpha_nv_n$. Since α_i is a real number, there exists some integer m_i' such that $\alpha_i + m_i' \in [0, 1)$. The point $m_1'v_1 + m_2'v_2 + \cdots + m_n'v_n$ is a lattice point. So we have

$$\begin{aligned}
\pi(x) &= \pi(\alpha_1v_1 + \alpha_2v_2 + \cdots + \alpha_nv_n) \\
&= \pi(\alpha_1v_1 + \alpha_2v_2 + \cdots + \alpha_nv_n) + 0_{\mathbb{R}^n/L} \\
&= \pi(\alpha_1v_1 + \alpha_2v_2 + \cdots + \alpha_nv_n) + \pi(m_1'v_1 + m_2'v_2 + \cdots + m_n'v_n) \\
&= \pi((\alpha_1 + m_1')v_1 + \cdots + (\alpha_n + m_n')v_n).
\end{aligned}$$

Since $(\alpha_1 + m_1')v_1 + \cdots + (\alpha_n + m_n')v_n$ is a point contained in T, $\pi(x) \in \mathrm{Im}(\pi\mid_T)$. Thus we have that $\pi\mid_T$ is a bijection, and so T is a tile. So the volume of T is the covolume of the lattice.

Let T' be any connected bounded tile. Since any tile is a transversal of $\mathbb{R}^n/L$, it must be the case that each point in T' is a translate of some point of the parallelepiped T. As T' is bounded, it must be the case that T' is contained in some ball B of finite radius r. Let $A = \{g_1, g_2, \ldots, g_t\}$ be the set of all lattice points such that $T + g_1$ intersects B. Since L is a discrete subgroup, A must be a finite set. Each point $x' \in T'$ may be expressed as

$x' = x + g_j$ for some $g_j \in A$ and $x \in T$. As T' is connected, the subset

$$T'_i = \{x' \in T' \mid x' = x + g_i, x \in T\} \subseteq T'$$

is also connected. As $f_i : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_i(x) = x - g_i$ is an isometry, it preserves volume. So the volume of T'_i is equal to the volume of the set $f_i(T'_i)$. The set $f_i(T'_i) \cap f_{i'}(T'_{i'}) = \emptyset$ if $i \neq i'$. So $T = \cup_{j=1}^{t} f_j(T'_j)$. This gives us that

$$\operatorname{Vol} T = \operatorname{Vol}\left(\cup_{j=1}^{t} f_j(T'_j)\right) = \sum_{j=1}^{t} \operatorname{Vol}\left(f_j(T'_j)\right) = \sum_{j=1}^{t} \operatorname{Vol}\left(T'_j\right) = \operatorname{Vol}\left(T'\right)$$

So any connected and bounded tile of the lattice L must have volume equal to the covolume of the lattice L. $\square$

Now we use this to define the density of a lattice covering.

Definition 1.11. The *density of the lattice covering* by a convex body S is the ratio of the volume of S to the covolume of the lattice. We will use $\delta_{L,S}$ to denote the density of the lattice covering obtained from the lattice L and the convex body S. If it is sufficiently clear from context, we will simply use δ to denote the density of a lattice covering. Any covering will have density of at least 1.

Let us return to some of the lattice coverings that we have previously seen. The first will be that of the lattice $\mathbb{Z}^2$ which was covered by disks of radius $\frac{1}{\sqrt{2}}$. We will use S_1 to denote the disk of radius $\frac{1}{\sqrt{2}}$ centered at the origin. We note that the area of each disk will be $\frac{\pi}{2}$. Next we note that $\{(1,0),(0,1)\}$ generates $\mathbb{Z}^2$. Thus the determinant of the matrix $\begin{bmatrix} e_1 & e_2 \end{bmatrix}$ is 1. So the density of this lattice covering is $\delta_{\mathbb{Z}^2,S_1} = \frac{\pi}{2}$ or about 1.57.

Next we look at the covering resulting from a connected and bounded tile T of a given lattice L'. The measure of any connected and bounded tile will always be the covolume of the lattice for which it is a tile. Thus regardless of our choice of lattice L', $\delta_{L',T} = 1$ when T is a tile of L'.

Corollary 1.12. *If a convex body S gives a lattice covering of $\mathbb{R}^n$ with the lattice L, then $\delta_{L,S} \geq 1$. If $\delta_{L,S} = 1$, then S is the closure of a tile.*

Proof. Let S be a convex body that gives a lattice covering with the lattice L. S must contain a tile by Theorem 1.6. As a tile is contained in S, $\operatorname{Vol} S$ is at least $\det \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$. Since $\delta_{L,S}$ is the ratio of the volume of the convex body to the covolume of the lattice, $\delta_{L,S} \geq 1$. If $\delta_{L,S} = 1$, then we see that $\operatorname{Vol} S = \operatorname{Vol} T$. As S is closed, the closure of S must be the set S. Now we note that the set T is contained in S. Let $\overline{T}$ be the closure of T. Then the set $S \setminus \overline{T}$ is open in S. If $S \setminus \overline{T}$ is not the empty set, then there exists some $x \in S$ such that there is an ϵ-ball about x that lies in $S \setminus \overline{T}$. This has a positive measure of volume, and so $S \setminus \overline{T}$ must have positive measure of volume. This contradicts the fact that $\delta_{L,S} = 1$, so $S \setminus \overline{T} = \emptyset$. Thus $S = \overline{T}$. $\square$

The lattice-simplex covering problem. While we may always find a way to cover $\mathbb{R}^n$ using a tile of a lattice L, a more interesting problem is that of using a convex body that is not a tile [1]. One such problem is that of finding a lattice covering using an n-simplex as the convex body, also called the *lattice-simplex covering problem*.

Definition 1.13. An *n-simplex* is a convex body in $\mathbb{R}^n$ that is defined by $n+1$ points in general position. In other words, it is a convex body that has a positive measure of volume in n-space that is defined by $n+1$ distinct points. The n-simplex is the smallest convex body that contains these $n+1$ points.

While the question is what choice of lattice and simplex will result in the lowest covering density, it has been shown that we may choose and fix the type of simplex and vary the lattice to find the optimal covering density [2]. We will choose the right-regular simplex to be the convex body used in our covering. In other words, our n-simplex will be a scaling of the convex body defined by the points $\{0, e_1, e_2, \dots, e_n\}$. Now we must choose a lattice that will result in the minimal density. One method of solving this problem is by choosing a lattice and looking at the tile that results from the subtraction construction. If this tile is contained in our right regular simplex, we know we have a covering. We also note that since the volume of a tile and the covolume of a lattice are equal, the density will be the ratio of the volume of the right-regular simplex to the volume of the tile. Recalling that the subtraction construction results in a minimum Manhattan diameter for any tile of our lattice, our choice of lattice will determine our tile. So we must choose the lattice that results in the volume of the tile from the subtraction construction being closest to the volume of a right regular simplex scaled by the Manhattan diameter of the tile.

This problem has been solved for $n < 3$. For $n = 1$, the 1-simplex will be an interval. So we may cover the space with a density of 1. For $n = 2$, the 2-simplex will be an isosceles right triangle, and the tile will be a rectangle or an L-shape. It has been shown that the best density for any rectangular tile is 2. It has also been shown that the best density for an L-shaped tile is 1.5. So we may cover $\mathbb{R}^2$ with a minimum density of 1.5.

For $n = 3$, the problem is still an open problem. It has been shown that the density will be less than 2. However, unlike the previous examples, there are infinitely many types of tiles that may result from the subtraction construction. This is part of the reason that the problem is still open for $n \geq 3$, as we cannot break the problem into a finite number of cases.

1.3.1 Multilattice problem. Now that we have set up the problem of finding the best density with a given lattice and simplex, we extend this to multilattices.

Definition 1.14. Let D be a finite subset of $\mathbb{R}^n$ which includes the origin. A *multilattice* is the set $\{g + d \mid g \in L, d \in D\}$. We write this set as $L + D$. We refer to L as the *base lattice*.

We often refer to the points contained in the set D as *displacements*, as they shift or "displace" the base lattice. In order to see what multilattices look like, we give a few examples. One example of a multilattice that is prevalent is that of the *edge-centered cubic lattice*, pictured in Figure 1.7. The displacement points are colored red, and the base lattice points are colored black. This is a scaling of the multilattice $2\mathbb{Z}^3 + \{0, e_1, e_2, e_3\}$. The reason that this is referred to as a lattice, despite not being a subgroup of $\mathbb{R}^n$, has to do with the

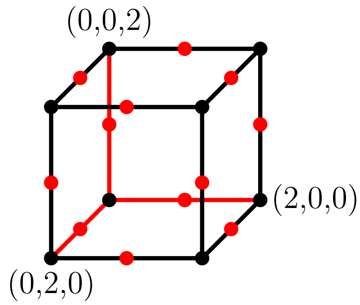

Figure 1.7: Edge-centered Lattice

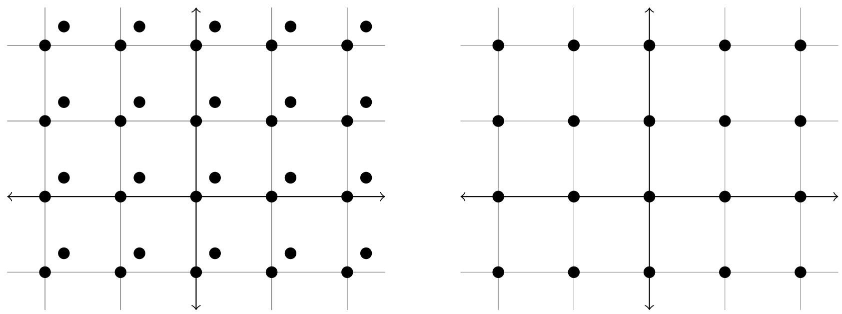

Figure 1.8: The multilattice $\mathbb{Z}^2 + \{0, (\frac{1}{4}, \frac{1}{4})\}$ and base lattice $\mathbb{Z}^2$

fact that it is frequently seen in crystalline structures in physics. In physics, a lattice is a physical model that may be described by using a periodic pattern of discrete points, and so the class of objects that physicists would describe as a lattice includes multilattices. To see that the edge-centered cubic lattice is a multilattice but not a lattice, note that it cannot form a group under addition. Add $(0, 1, 0)$ and $(1, 0, 0)$, which are both in the edge-centered lattice. We get the point $(1, 1, 0)$, which is not included in the edge-centered cubic lattice.

Another example is that of the lattice $\mathbb{Z}^2$ together with the image of the translation $f(x) = x + (\frac{1}{4}, \frac{1}{4})$. This multilattice is pictured next to the base lattice in Figure 1.8.

As we work with multilattices, it is worth noting there may be multiple displacements that get us the same multilattice. For example, let us take our base lattice to be $\mathbb{Z}^2$ once again. Then let $\{0, (\frac{1}{3}, \frac{1}{3})\} = D$. The displacement will result in the set $\{(\frac{3x+1}{3}, \frac{3y+1}{3}) \mid x, y \in \mathbb{Z}\}$ being in the multilattice $L + D$. Now suppose instead we had chosen the displacement $(\frac{4}{3}, \frac{4}{3})$. The set from this displacement is $\{(\frac{3x+4}{3}, \frac{3y+4}{3}) \mid x, y \in \mathbb{Z}\}$. These both define the same set of points. For this reason, we may choose a point that lies on the interior of our tile from

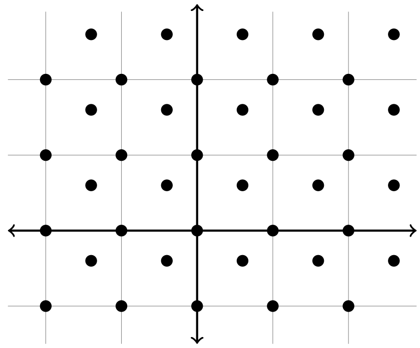

Figure 1.9: $\mathbb{Z}^2 + \{0, (\frac{3}{5}, \frac{3}{5})\}$

the subtraction construction whenever it is convenient.

With the new object of a multilattice, we note that if we place convex bodies at each point, we may still get a covering, but not necessarily a lattice covering.

Definition 1.15. Let $D = \{d_0 = 0, d_1, d_2, \ldots, d_k\}$ be a subset of $\mathbb{R}^n$ and C_j $(0 \le j \le k)$ be a convex body such that $d_j \in C_j$. A *multilattice covering* of $\mathbb{R}^n$ by the set of convex bodies $\{C_j \mid 0 \le j \le k\} = \mathcal{C}$ is a covering such that $\mathbb{R}^n = \cup_{j=0}^{k}(C_j + L)$. The convex bodies placed at two distinct displacements need not be translates of each other.

Now that we have a method of covering using a multilattice, we also define the density of such a covering.

Definition 1.16. The *density of a multilattice covering* is defined to be the ratio of the sum of the volume of each of the convex bodies C_i to the covolume of the base lattice L. We will denote this as $\delta_{L+D,\mathcal{C}}$ for the multilattice covering of $\mathbb{R}^n$ generated by the multilattice $L + D$ and the convex bodies $\mathcal{C} = \{C_i\}$. If it is sufficiently clear from context, we will use δ to denote the density of a multilattice covering.

We now look at an example of a multilattice covering, and find the multilattice covering density. Let us take the lattice $\mathbb{Z}^2$ as our base lattice. Let our set of displacements be $D = \{0, (\frac{3}{5}, \frac{3}{5})\}$ Then we get the multilattice in Figure 1.9. We note that under the subtraction construction of the base lattice, we get a square of side length 1 for our tile. So we may choose a set of convex bodies that will contain this tile to get a covering. Let us

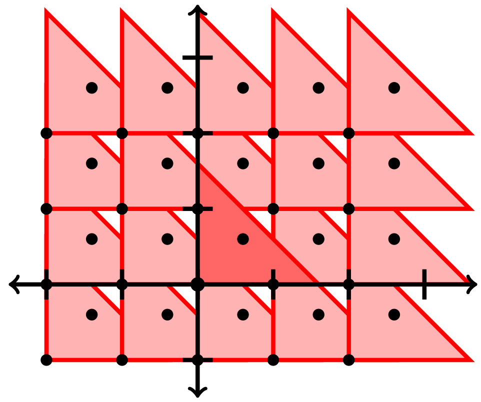

Figure 1.10: $\mathbb{Z}^2 + \{0, \left(\frac{3}{5}, \frac{3}{5}\right)\}$ partial covering by simplices at lattice points

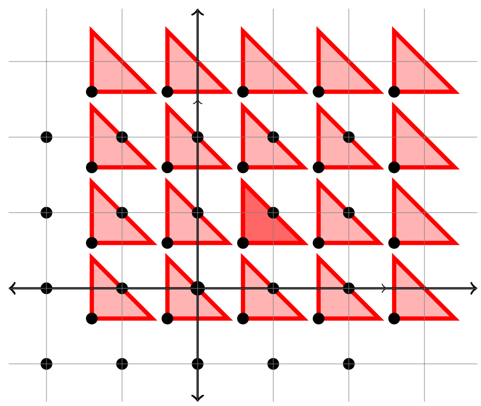

Figure 1.11: $\mathbb{Z}^2 + \left\{0, \left(\frac{3}{5}, \frac{3}{5}\right)\right\}$ partial covering by simplices at displacement points

place the right regular simplex of side length $\frac{8}{5}$ at each point in our base lattice. Then we get the non-cover in Figure 1.10. Now let us also place the right regular simplex of side length $\frac{4}{5}$ at each of our displacement points. We draw this result in Figure 1.11 where we only place the simplices at each displacement point. With just the simplices placed at the displacement points, we still don't have a covering. However, by placing the right regular simplex of side length $\frac{8}{5}$ at the base lattice points, and the right regular simplex of side length $\frac{4}{5}$ at the displacement points, we do obtain a multilattice covering, pictured in Figure 1.12. As $\mathbb{Z}^2$ has a covolume of 1, this covering has a density equal to the sum of the area of each simplex used in our multilattice covering, or $\frac{32}{25} + \frac{8}{25} = \frac{8}{5}$. With the base lattice of $\mathbb{Z}^2$, the subtraction construction resulted in a lattice covering with a density of 2. So given a multilattice, we may find coverings that result in a lower density than the base lattice covering using right

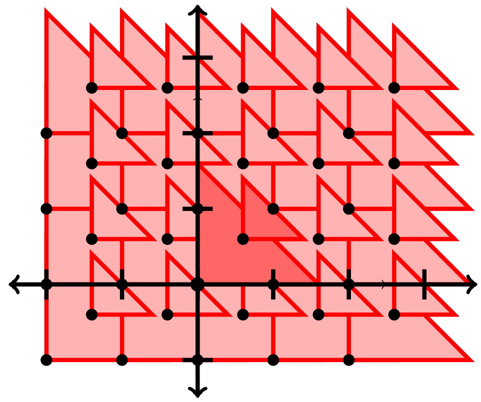

Figure 1.12: Multilattice covering with right regular simplices

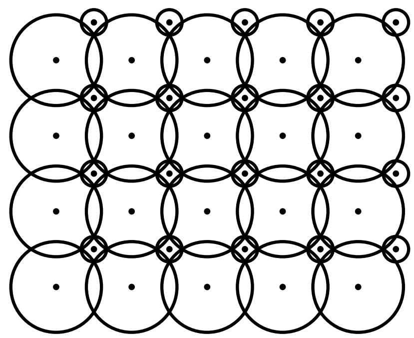

Figure 1.13: Multilattice covering with disks

regular simplices.

We now do one more example of finding the multilattice covering density of a given multilattice covering. Let us take the lattice $\mathbb{Z}^2$ once more. Then let $D = \{0, \left(\frac{1}{2}, \frac{1}{2}\right)\}$. For the convex body that is placed on the points of our base lattice, take the disk of radius 0.6 centered on each point. Then let the convex body for our displacement be a disk of radius $\sqrt{0.11}$ centered on each displacement. This covering gives us a density of $\pi\left(\frac{72}{100} + \sqrt{\frac{11}{100}}\right)$, or about 1.2199983.

Our question now is, given k displacements, what is the best possible density for a multilattice covering that uses dilations of the right regular simplex at each point?

Chapter 2. Two Displacements in Two Dimensions

We address the problem of finding the best multilattice covering density first by limiting ourselves to two displacements and two dimensions. As we will be using dilations of the right regular simplex to generate our coverings, we will simplify the problem by standardizing how we may place these convex bodies to generate a covering.

Definition 2.1. We will say we are *placing a simplex S at a point p* when the vertices of our simplex are the set of points $\{p, p + ae_1, p + ae_2, \ldots, p + ae_n\}$ where a is a positive real number. We will also say this simplex has *scale a*.

Now we turn to the problem of placing simplices such that the density of a multilattice covering is minimized. When $n = 1$, we may obtain a lattice covering with a density of 1. So we begin with multilattice coverings when $n = 2$. Given a lattice L, we know that by the subtraction construction we get a given density δ_L. However, if we add a single displacement, and we place dilations of the right regular simplex at each point in our multilattice, what is the new density we can achieve?

We start by listing a few results from our lattice coverings. The best lattice covering in two dimensions has a density of 1.5. There are two types of tiles that result from the subtraction construction when $n = 2$. The two are the rectangular tile and the L-shaped, or 2-staircase, tile [2].

Lemma 2.2. *There exists a multilattice covering by the multilattice $L+D$ using only dilations of the right regular simplex of at most density δ_L.*

Proof. The density of the lattice covering of the base lattice is δ_L. Let T be the tile obtained from the subtraction construction. Then let S be the simplex of scale $\operatorname{Mdiam}(T)$. Now let $L + D$ be a multilattice with $|D| = k + 1$. Then we must choose a collection of right regular simplices S_i for $0 \leq i \leq k$ such that when S_i is placed at d_i for each $d_i \in D$ we will get a multilattice covering of $\mathbb{R}^n$. Additionally, we require $\sum_{i=0}^{k} \mathrm{Vol}\,(S_i) \leq \mathrm{Vol}\, S$. Let S_0 be the right regular simplex of scale $\mathrm{Mdiam}\,(T)$ placed at d_0. Now let us choose S_i to be the point d_i for $1 \geq i \geq k$. We note that this would be the right regular simplex of scale 0. Then the measure of n-volume of S_i will be 0 for $1 \geq i \geq k$. Notice that $\sum_{i=0}^{k} \mathrm{Vol}\,(S_i) = \mathrm{Vol}\, S + \sum_{i=1}^{k} \mathrm{Vol}\, S_i = \mathrm{Vol}\, S + \sum_{i=1}^{k} 0 = \mathrm{Vol}\, S$. We also note that S_0 is the right regular simplex that resulted in a lattice covering. So there exists some tile T such that T is contained in S_0. Thus T is also contained in $\cup_{i=0}^{k} S_i$. So this collection of simplices provides a multilattice covering with density δ_L. □

So we assume that the density for any multilattice covering is at most δ_L, where L is the base lattice. It may be possible to obtain a multilattice covering with a density δ_{L+D} such that $\delta_{L+D} < \delta_L$ with the proper choice of S_i. We have seen the example of the multilattice covering of $\mathbb{Z}^2 + \{0, \left(\frac{3}{5}, \frac{3}{5}\right)\}$, pictured in Figure 1.12, has a covering density lower than that of the lattice covering given by $\mathbb{Z}^2$. The best density we may achieve for a given multilattice depends on our set of displacements D. We now focus on how we may choose the base lattice and add a single displacement that will result in the best possible covering density. We also note that depending on what displacement we choose with our base lattice, we will need different sized simplices for both the base lattice points and the displacement points to cover the tile from the subtraction construction. So we now explore how we may decrease the total area of these simplices while still covering the tile. As the tile from the subtraction construction will only depend on the base lattice, our choice of displacement will not affect the tile type.

2.1 Simplifications

Some assumptions may be made as in the lattice covering problem to simplify the question. We first start by noting that a lattice and simplex used in a covering may be simultaneously scaled without changing the density. We will use this to scale our lattice and simplex in such a way to give the tile T a certain value for its Manhattan diameter. If we allow for any simplices to be chosen for our multilattice covering, we now show that we can achieve a density of 1 in $\mathbb{R}^n$ for any n with a finite number of displacements. To see this, simply take the lattice $\mathbb{Z}^n$. The unit cube is comprised of a finite number of simplices whose vertices are the vertices of the cube, and whose intersection with each other is an m-simplex, with $m < n$. So we simply choose these simplices as our convex bodies, and we choose the displacements to be one of the vertices of a given simplex. This will be the unit cube, which is the closure of a tile of $\mathbb{Z}^n$. Now let us restrict ourselves to a multilattice covering where each simplex used in the covering is a dilation of a single simplex which contains the origin. In other words, each simplex in our covering is the same shape, but may be a different size. We note that this cannot result in a covering density of 1 if $n > 1$. We now explore what the lowest multilattice covering density using a single type of simplex is. For convenience we choose the right regular simplex.

2.2 Ensuring the efficiency of a multilattice covering

Performing the subtraction construction with a lattice L gives us a tile that lies inside the right regular simplex S such that $L + S = \mathbb{R}^n$ and if S' is any right regular simplex with volume less than the volume of S, $S' + L \neq \mathbb{R}^n$. In other words, the Manhattan diameter of the tile obtained from the subtraction construction is the minimum Manhattan diameter of any tile. We now need to worry about choosing the correct dilation of the right regular simplex for each multilattice point to ensure that the multilattice covering also has minimum density. A *tessellation* of a tile T is any partition of T. The tile obtained from the subtraction construction may be broken up into a tessellation by following a process similar to the subtraction construction, formally outlined in Theorem 2.7. We start this process by taking a tile T obtained from the subtraction construction, rather than an orthant, and take pieces out of it one at a time based on the multilattice points contained in T. As there are only finitely many points that we may use in this process, we will only have finitely many pieces of our tile. These pieces will become what we refer to as prototiles of our tessellation.

Definition 2.3. If a given tile T of a lattice L is divided into finitely many mutually disjoint subsets $T_1, T_2, \ldots, T_k$, then each T_i is referred to as a *prototile* of T.

A prototile may be thought of as a tiling of a tile, in that a set of prototiles will cover a tile with no overlap, just as a tile covers a space with no overlap. We may define a tessellation of a tile by defining the set of prototiles of the tile. By choosing a set of prototiles of our tile T from the subtraction construction, we may define a multilattice covering by covering each prototile. If the correct set of prototiles is used, we minimize the multilattice covering density. We will now formalize this process.

Lemma 2.4. *Let a multilattice covering of $\mathbb{R}^n$ be given. Let T be a tile obtained from the subtraction construction. There exists a tessellation of T into k prototiles that are the union of sets of the form $(F + y) \setminus (F + P)$ for some $y \in T$ and set of points $P \subseteq L + D$. which are covered by k distinct dilations of the right regular simplex.*

Proof. Let us take a multilattice covering of $\mathbb{R}^n$ by right regular simplices. Let $L + D$ be the multilattice. If $|D| = k$, then there will be k distinct dilations of the right regular simplex. Let T be a tile obtained from the subtraction construction. Any displacement d may be replaced by $d + g$, where g is an element of the base lattice, and still give the same multilattice. We choose a collection of simplices A such that T is covered by the simplices in the collection. We now take the set of multilattice points that these simplices are placed at and call this set D''. We note that D'' must exist, as T is a subset of $\mathbb{R}^n$. We call the point the simplex S_i is placed at d''_i We also note that $|D''| < \infty$, as the covering density is finite. T must have a corner $u_1 = (a_1, \ldots, a_n)$ such that $\sum_{i=1}^{n} a_i = \mathrm{Mdiam}\,(T)$. If there are multiple points with this property take the corner that would be first under the lexicographical order. Take the set of simplices which contain u_1, and call this set $A_1 = \{S_i | u_1 \in S_i\}$. Let the set $B_1 = \{s_i \in L + D \mid S_i \text{ is placed at } s_i, S_i \in A_1\}$.

Take the simplex S_j in A_1 such that $\mathrm{Mdiam}\,(u_1, s_j \geq \mathrm{Mdiam}\,(u_1, s_i)$ for every $s_i \in B_1$. If there are multiple simplices with this property, take the simplex placed at the s_j first under the lexicographical order. Let $s_j = d''_j$. Then we note that the region $T \cap (d''_j + F)$ is contained in S_j. Call this region T_1. Let $T_1' = T \setminus T_1$. Now let $D_2'' = D'' \setminus \{d_j''\}$. Now we note that T_1' has a point u_2 such that $\mathrm{Mdiam}\,(0, u_2) = \mathrm{Mdiam}\,(T_1')$. If there are multiple, we once again take the point first under the lexicographical order. Now take the set A_2 of all simplices which contain u_2. Let the set $B_2 = \{s_i \in L + D \mid S_i \text{ is placed at } s_i, S_i \in A_2\}$. Take the simplex S_k in A_2 such that $\mathrm{Mdiam}\,(u_2, s_j \geq \mathrm{Mdiam}\,(u_2, s_i)$ for every $s_i \in B_2$. If there are multiple simplices with this property, take the simplex placed at the s_j first under the lexicographical order. Now let $T_2 = T_1' \cap (d_k'' + F)$ and $T_2' = T_1' \setminus T_2$. Let $D_3'' = D_2'' \setminus \{d_k''\}$. We continue this process until $T_{m''}'$ is the empty set. Each of the T_i is disjoint from any other. We also note that $\cup_{i=1}^{m'} T_i = T$, and so we have a tessellation of T. Each of the T_i must be a set of the form $(F + y) \cap (F + P)$, as T_i may be expressed as $T \cap ((F + d_i'') \setminus (F + \{d_1'', d_2'', \ldots, d_{i-1}''\})$. So we will have a point $y \in T$ such that $(F + y) \cap T = (F + d_i'') \cap T$. However, we note that $m' \geq k$. If $m' = k$, then we are done, as there will be one prototile for each displacement. Suppose that $m' > k$. Then we note that as there are only k choices for a distinct dilation of a simplex, at least 2 of the simplices are placed at points that differ by a lattice element. That is, if d_i'' and d_j'' are the points that these two simplices are placed at, then $d_i'' - d_j'' \in L$. We also note that the two simplices must have the same area. Thus, these two simplices map to the same set under the projection map. So our choice of d_i'' or d_j'' in D'' will get us the same multilattice. Let D_2'' be the set $D'' \setminus \{d_i''\}$. Now if $|D_2''| \neq k$, we continue. Eventually we will have a set D_i'' such that $|D_i''| = k$. Additionally, $L + D = L + D_i''$. If T_i and T_j are regions covered by 2 simplices that are not distinct dilations, then $T_i \cup T_j$ must be covered by the same dilation of the right regular simplex. Thus, the region covered by each distinct simplex is $\cup_{i=1}^{\ell} T_{m_i}$, where $\{m_1, \ldots, m_\ell\}$ is the subset of D'' such that $d_{m_1}'' = d_{m_j}'' + g$ for some lattice element g. □

So we now know that any covering must give a tessellation of the tile T. This must include the multilattice covering that gives the minimal covering density. So we know that any multilattice covering may be thought of in terms of how it divides T into prototiles. So we turn our attention now to how we may approach the problem of breaking T into prototiles in such a way as to give the minimal covering density. We start with how we may find a cover given a multilattice.

Lemma 2.5. *Let the simplices $S_0, S_1, \ldots, S_k$ be the simplices used to generate a multilattice covering of $\mathbb{R}^n$ for a multilattice $L + D$. If S is the simplex with minimal volume such that $T \subseteq S$, where T is the tile obtained from the subtraction construction, then we may replace any S_i in our multilattice covering with a simplex with volume $\operatorname{Vol} S$ and still retain a cover.*

Proof. If $\operatorname{Vol} S_i \leq \operatorname{Vol} S$, then every point contained in S_i is also contained in the simplex S when placed at the same point. Suppose that S_i has n-volume greater than S for some i. S_i must be placed at some displacement d. Without loss of generality, let us assume that $d \in T$. Let us take the region of $\mathbb{R}^n$ obtained by shifting T by d. This is still a transversal of all cosets of $\mathbb{R}^n/L$. This must also be a subset of the region obtained by shifting S by d. This must be a subset of S_i placed at d, as both simplices are a dilation of the right regular simplex. So S placed at d contains a tile. Thus we still have a covering if we replace S_i with the simplex of size S. $\square$

Theorem 2.6. *There is no multilattice covering of $\mathbb{R}^n$ for a multilattice $L + D$ with density less than δ_L for any simplex which covers a prototile of T and is placed at a displacement with any coordinate whose value is greater than* $\operatorname{Mdiam}(T)$.

Proof. We begin by taking a multilattice covering. Let $\{S_0, S_1, \ldots, S_k\}$ be the set of distinct simplices that are used in this cover. Using Lemma 2.5 we replace any simplices with area greater than that used in the lattice covering with the simplex S used in the lattice covering. S must have the same Manhattan diameter as the tile T. Now let d be any displacement such that if $d = (x_1, x_2, x_3, \ldots, x_n)$ there exists some i such that $|x_i| > \operatorname{Mdiam}(T)$. We note that any simplex placed at d must have Manhattan diameter at most $\operatorname{Mdiam}(T)$. If $x_i < 0$, then $x_i + \operatorname{Mdiam}(T) < 0$. We note that any point with a negative coordinate must lie outside T, so the simplex at d cannot intersect with T. If $x_i > 0$, then we note that for any point $y = (y_1, \ldots, y_n)$ in T, $x_i > y_j$ for every j. As any point in the simplex placed at d will have ith coordinate $x_i + c$, where $c \geq 0$, there is no point of T that lies in this simplex. So the simplex S placed at d has an empty intersection with T, and there cannot be a prototile contained in the simplex placed at this point. $\square$

We know that we need only look at a finite number of displacements of a given multilattice, and worry about how to determine what the least dense cover possible will be. To do this we will take the finite set of displacements we need worry about and we will use a method similar to the subtraction construction.

The algorithm below finds a minimal density covering for a given multilattice in $\mathbb{R}^n$.

Theorem 2.7. *The following steps will result in a minimum covering density for a given multilattice $L + D$.*

(i) Find all points of $L + D$ that lie in the n-cube of side length $2\,\mathrm{Mdiam}\,(T)$. Call this set A

(ii) Let $D' = \{d'_i\}$ be the representatives of D that lie on the interior of T. Let $A_i = \{x \in A \mid x = d'_i + g, g \in L\}$.

(iii) Define T'_1 to be the tile from the subtraction construction.

(iv) The region T'_j must contain a point $(a_1, \ldots, a_n)$ such that $\sum_{i=1}^n a_i \geq \sum_{i=1}^n x_i$ for any $(x_1, \ldots, x_n)$ in T'_j. In case there are multiple such points, take the point first under the lexicographical order.

(v) Now for each set A_i, we take the point $y = (y_1, \ldots, y_n)$ in A_i such that $a_i - y_i > 0$ and $\sum_{i=1}^n a_i - y_i \leq \sum_{i=1}^n a_i - y'_i$ for any $(y'_1, \ldots, y'_n)$ in A_i. Place a simplex of scale $\sum_{i=1}^n a_i - y_i$ at the point y and call it S_j. This simplex must cover $(a_1, \ldots, a_n)$ with minimum possible volume when placed at y.

(vi) We now define the prototile that is covered by this simplex. Take the set of points $T'_j \cap (F + y)$ and call it T_j. This set must be $(F + y) \setminus (F + P)$ for some set P, and is contained in S_j.

(vii) *Now we define* T'_{j+1} *to be* $T'_j \setminus T_j$. T'_{j+1} *is the region that is not covered by* $S_{j'}$ *for* $j' \leq j$.

(viii) *We repeat steps* $(iv) - (vii)$ *in the same way. We continue this process until* T'_{j+1} *is the empty set.*

(ix) *We now take all simplices that were placed at points of* A_i *for each* i. *Let* S_m *be the largest such simplex. Replace each simplex in this collection by a simplex of the same scale as* S_m.

(x) *We do this for each* $i \leq |D|$, *and have* $|D|$ *distinct simplices.*

(xi) *Let* S'_i *be the simplex placed at the displacement* d'_i. *Then* $\{S'_i \mid 0 \leq i \leq k\}$ *is the set of distinct simplices required to cover* $\mathbb{R}^n$. *Now we note that the density is given by*

$$\delta = \frac{\mathrm{Vol}\, S_0 + \mathrm{Vol}\, S_1 + \cdots + \mathrm{Vol}\, S_n}{\det\{L\}}.$$

(xii) *Take a covering with the minimum density in this collection.*

Proof. We note that the above process will generate all possible prototiles that are unions of sets of the form $(F + y) \setminus (F + P)$, with $P \subseteq L + D$. This is because each time we cover a region of T'_i, we are defining a prototile that is of the form $(F + y) \setminus (F + P)$. As we check each displacement as the cover, we will get all possible divisions of T into sets of the form $(F + y) \setminus (F + P)$ covered by the displacements in the n-cube of side length $2\,\mathrm{Mdiam}\,(T)$. By Theorem 2.6, we know that one of these will have the minimum covering density of any tessellation of T by sets of the form $(F + y) \setminus (F + P)$. We also note that by Lemma 2.4 the minimum possible covering density of $L + D$ may be defined using sets of the form $(F + y) \setminus (F + P)$. If π is the standard quotient map, notice that $\pi(S_k) \subseteq \pi(S_m)$ where S_k and S_m are placed at the same point. Thus, the minimum density obtained in this way will be the minimum possible density of δ_{L+D}. □

We notice that the prototiles of this covering are defined in the same way as the pieces we remove in the subtraction construction. We also note that any covering may be defined this way. Thus we may consider the problem geometrically as the most effective way of covering the tile T. We no longer worry about how to choose what simplex we place at a given displacement. Now we turn our attention to what choice of multilattice will lead to the lowest covering density.

2.3 Selecting the right displacement

Now that we have a process for choosing simplices that result in a covering for a multilattice, we find the best displacement points possible. The first step we must take is determining where to place our displacement relative to the tile T. We begin with the problem of covering $\mathbb{R}^2$. We begin by proving that with a single displacement, we need only consider the simplex placed at a displacement inside T.

Theorem 2.8. *For any multilattice $L + D$ in $\mathbb{R}^2$ with $|D| = 2$ and multilattice covering density δ_{L+D}, there exists a multilattice $L' + D'$ such that the tile from the subtraction construction on L' may be covered by only two right regular simplices and $\delta_{L+D} = \delta_{L'+D'}$.*

If we show this, then we may reduce finding the multilattice $L + D$ with the minimum density to the question of how to cover a 2-staircase with only 2 right regular simplices. We do not need to worry about the case where a prototile of T is the union of 2 n-staircases.

Proof. Assume by way of contradiction that there exists some multilattice $L + D$ such that there is no $L' + D'$ with $\delta_{L+D} = \delta_{L'+D'}$ and the tile of L' from the subtraction construction has 2 prototiles that are n-staircases. The tile T obtained from the subtraction construction on L is a 1-staircase or a 2-staircase. The density $\delta_{L+D} < 1.5$, otherwise we could just use a lattice covering with the same density, and the simplex of area 0 at the displacement. We will also define the rectangle from (a, b) to (c, d) to mean the rectangle with vertices $(a, b), (c, b), (c, d), (a, d)$. We now break this into cases.

We begin with the case that T is a 1-staircase. Let the outermost corner be at (a, d). If T must be covered by at least 3 simplices to obtain a given density, and there are only

2 distinct simplices, it must be the case that at least two of the simplices are translates by some $g \in L$. We may choose $(x, y) \in D$ such that $x < a$ and $y < d$, and $x > 0$ or $y > 0$. We must cover (a, d) with the simplex placed at (x, y). Without loss of generality, assume that $x > 0$. Now we note that when we take the region covered by the simplex placed at (x, y), one corner of prototile not covered by this simplex is (x, d). We also get a corner at (a, y) if $y > 0$.

Let us take the case with $y > 0$. We now have the two corners (a, y) and (x, d). If $x + d \geq a + y$, then we note that (x, d) cannot be covered by the simplex placed at $(0, 0)$, otherwise $L + D$ would fill the requirements of $L' + D'$. So (x, d) is covered by a displacement at $(x', y') = (x, y) + g$. Then we get another 1-staircase contained in T which is contained in the simplex placed at (x', y'). The prototile obtained from the region covered by these two simplices forms the shape of a 2-staircase reflected about the origin.

The point (a, y) must also be covered by a simplex. If it is covered by the simplex at the origin, then the base prototile is 2-staircase. As a right simplex must have a ratio of at least 1.5 to a 2-staircase it contains, both simplices must have an area of at least 1.5 times the area of the prototile they contain. Thus our density is at least 1.5, a contradiction. So (a, y) must be covered by the displacement at $(x, y) - g'$. The simplex placed at (x, y) must have a scale of at least $(d + a - x)$. This breaks T into two regions, a 1-staircase, and a 2-staircase reflected about the origin. the ratio of area of the simplex to the area of the prototile must be at least 1.5 for both prototiles. This will once again give us a density of at least 1.5 for the multilattice covering. As we need the density to be less than 1.5, $y \not> 0$.

So $y = 0$, as we may choose a representative that lies in T. So the region covered by the simplex placed at (x, y) must be a 1-staircase, and the remaining region is a 1-staircase. The region not covered by the simplex at (x, y) will have corner (x, d). (x, d) must be contained in a simplex placed at $(x, y) + g$. If this is the case we have a prototile that is a 2-staircase reflected about the origin. The base prototile will be a 1-staircase. So the density must be at least 1.5 again.

So every lattice L with a rectangular tile from the subtraction construction has some multilattice $L' + D'$ with density at most δ_L and T' covered by two right regular simplices.

We now check the case where T is a 2-staircase. T must have two corners, call them (a, d) and (c, b). Without loss of generality, let $a + d \geq c + b$ and $a < c$. Now let $(x, y) \in T$ be the non-trivial displacement of D. If $x \geq a$, then to cover (a, d), the simplex must have scale at least $(c - x + a + b - y)$. So we get a rectangular section of T covered by the simplex placed at $(x, y) + g$ for some g in L, where the rectangular section is defined by corners $(0, d + y - b)$ and (a, d). However, note that we may shift the rectangle $(0, d + y - b)$ and (a, d) by $-g$, and the new shape will still be a tile T'. T' is the union of a 3-staircase contained in T, with corners $(a, d + y - b)$, (x, b), and (c, y); and the rectangle (x, y) to $(c + a, b)$. As the area of a right regular simplex to the area of a 3-staircase contained in the simplex may have density less than 1.5, we may generate a multilattice covering of $\mathbb{R}^2$ by covering T' with two simplices whose areas sum to less than $1.5(\mathrm{Vol}\,(T'))$. Note that the ratio of the volume of the simplices to the volume of T' may be decreased by decreasing the area of the rectangle (x, y) to $(c + a, b)$. We do this by decreasing the x-coordinate of our rectangle.

If $c - x \geq b - y$, then we will decrease the density of our multilattice covering by decreasing the rectangular prototile to (x, y) to (c, b). In this case our density will also decrease, and we see that this tile T' will be a 2-staircase. If $b - y > c - x$, then we will reduce our rectangular prototile to (x, y) to $(b - y + x, b)$. This will be a square. We now turn back to the 3-staircase. By the above, $x + b > c + y$. If we change the x-coordinate of the corner (c, y) to $b - y + x$, then we will not change the Manhattan diameter of our 3-staircase. However, it will increase the area of T'. So we do this, and we again get a 3-staircase and 1-staircase as prototiles of T', with T' a 2-staircase.

In either case, T' may be covered by 2 right regular simplices. The ratio of the area of these 2 dilations of the right regular simplex to the 2-staircase will be less than δ_{L+D}. So $x \ngeq a$.

Let $x < a$. We once more place a simplex at (x, y) to cover (a, d). Then we note that the new corners of the 3-staircase must be (x, d), (a, y), and (c, b). We now see the cases where each corner determines the Manhattan diameter of the 3-staircase.

If $x + d > a + y$ and $x + d > c + b$, we note that we must once more cover (x, d) with a simplex placed at a displacement. The simplex placed at (x, y) must have scale at least $a + d - y + b$, and so to have better density than the lattice using L, $y > b$. So the simplex placed at $(x, y)+g$, $g \in L$ will only cover (c, b) if the simplex has scale at least $a-x+d-y+b$. If this is the case, then we note that there is a tile of L that is the union of the 2-staircase placed at $(0, 0)$ with corners (a, y) and $(c-a+x, b)$, and the 2-staircase placed at (x, y) with corners $(a, d + b)$ and $(x + a, d)$. This tile will have the same area as T. This tile will be covered by the two simplices place at $(0, 0)$ and (x, y) with the same area as the simplices used in the multilattice covering of $\mathbb{R}^2$ by $L + D$. However, the simplex to 2-staircase ratio of each section of the new tile must be at least 1.5. Thus, the density of $L + D$ would be at least 1.5. So $a + y > x + d$ and $a + y > c + b$ or $c + b > x + d$ and $c + b > a + y$.

If $c + b > x + d$ and $c + b > a + y$, then (c, b) must be covered by a simplex at a displacement. However, the displacement must have x coordinate $c - a + x$, and a negative y coordinate. So the simplex at this point will cover the rectangle $(c - a + x, 0)$ to (c, b). However, we may choose a lattice whose tile will have corners (a, d) and $(c - a + x, b)$. Now if our simplex at (x, y) has scale $x + y$, this will reduce the ratio of the simplex at (x, y) to the region of T not covered by our base simplex. This reduces the density of our multilattice covering by a lattice with the properties of $L' + D'$. So this will reduce to the previous case or the case where $a + y > x + d$ and $a + y > c + b$.

So it must be the case that $a + y > x + d$ and $a + y > c + b$. If (a, y) is covered by a displacement simplex, notice that our simplex must cover the rectangle $(x, y - b)$ to (a, d) or (x, b) to (a, d). If $b \geq y - b$, it will be the rectangle (x, b) to (a, d). If the displacement simplex has scale at least $d - y + a + b$, then a displacement simplex will also cover the rectangle $(0, b)$ to (x, d), and the base simplex must cover a 1-staircase. Thus there is a tile that is a union of a rectangular prototile and a 2-staircase prototile covered by the same set of simplices. So our density would be greater than 1.5. So it must be the case that (x, d) is not covered by a displacement. Note that there is a tile of L that is a union of 2 2-staircases covered by these simplices, and so our density δ_{L+D} is at least 1.5. So it must be the case that $b < y - b$.

This breaks into the cases of (x, d) being covered by the displacements or by the simplex at the origin. If it is the former, then there is a tile that is a union of 2 2-staircases once more, so the density is at least 1.5. So (x, d) is not covered by the displacements.

If (x, d) is covered by the base simplex, there is no equivalent tile that is a union of the 3-staircase covered by the simplex at the origin and a 2-staircase placed at (x, y). This is because we cover the rectangle $(x, y - b)$ to (a, d) with a displacement simplex. If we want to have a corner with x coordinate greater than a, the equivalent region must come from the rectangle with vertex $(0, y - b)$ but Manhattan diameter less than $x + d + b - y$. But this rectangle will lie in the interior of the region that must be covered by the base tile. Thus, if the density of a multilattice covering with $|D| = 2$ is less than 1.5, then the prototiles of T must be a 3-staircase and rectangle. As we have done previously, we may use a displacement d' to cover these two prototiles. We then obtain a multilattice covering with the rectangle $(x, y-b)$ to (a, d) covered by the displacement simplex, and the remaining region of T covered by the simplex placed at $(0, 0)$.

But this is every case, and so our assumption that that there exists a multilattice $L + D$ with density δ_{L+D} with no multilattice $L' + D'$ such that T' from the subtraction construction on L' may be covered by only 2 simplices and $\delta_{L+D} = \delta_{L'+D'}$ is false. □

With this we have that we need not worry about any case other than when we cover T with a simplex placed at the origin, and a simplex placed at a single displacement d. Notice that when we add a single displacement to a lattice, it will cover a rectangular section of the base tile, or an L-shaped section. Thus, we break the problem of finding the best multilattice with one displacement into these two cases. We use the following theorem to simplify this.

Theorem 2.9. *Any multilattice covering of $\mathbb{R}^2$ that has density less than 1.5 must have at least one connected region that is a rectangle. .*

Proof. To prove this theorem, we will note that the density of a multilattice covering may be thought of as the weighted average of the ratio of each simplex which covers a prototile to the area of the prototile. The weights in this average are the proportion of the area of the tile that the prototile covers. Now let us take any multilattice covering with a density $\delta_{L+D} < 1.5$. We note that at least one of the prototiles in this covering must have at least 3 outer corners, or must be a 3-staircase. We know this as at least one of the simplex area to prototile areas must result in a ratio less than 1.5, and a 2-staircase has at least a density of 1.5. We note that starting from the tile, we will have a 2-staircase or a rectangle. If the tile is a rectangle then we are done, as any prototile which contains the outermost corner cannot contain another outer corner, by Theorem 2.8. If the tile is a 2-staircase, then we note that if we do not have a rectangular prototile to cover the outermost corner, we must have a 2-staircase prototile. If we have a 2-staircase, then we note that at least 2 outer corners must be contained in it. However, our tile only has 2 corners, so a prototile must cover both of the outermost corners of our tile. If our prototile is the tile itself, we cannot have a density less than 1.5. So we assume that the prototile is not the whole tile. We must then have a positive coordinate for our displacement. If the second coordinate is nonpositive, then our prototile will be a 2-staircase, but the remainder of our tile must be a rectangle. Thus this will be a prototile, or from the above must contain a rectangular prototile.

So we must have a displacement with two positive coordinates. But we note that if d_1 is this displacement, and u_1 is the squinch, then $u_1 - d_1$ must be positive in each coordinate. We now know that $u_1 + F \subseteq d_1 + F$. We note that the part of the tile not contained will be the set of points $F \setminus ((L^{0+} + F) \cup (d_1 + F))$. As $u_1 - d_1$ is positive, d_1 will be the only blocker for the remainder of the tile that lies in the first quadrant. Thus the remainder of the tile must be a 2-staircase. As D is a finite set, this means that all prototiles must be 2-staircases if there is not a rectangular prototile. Thus the density must be a weighted

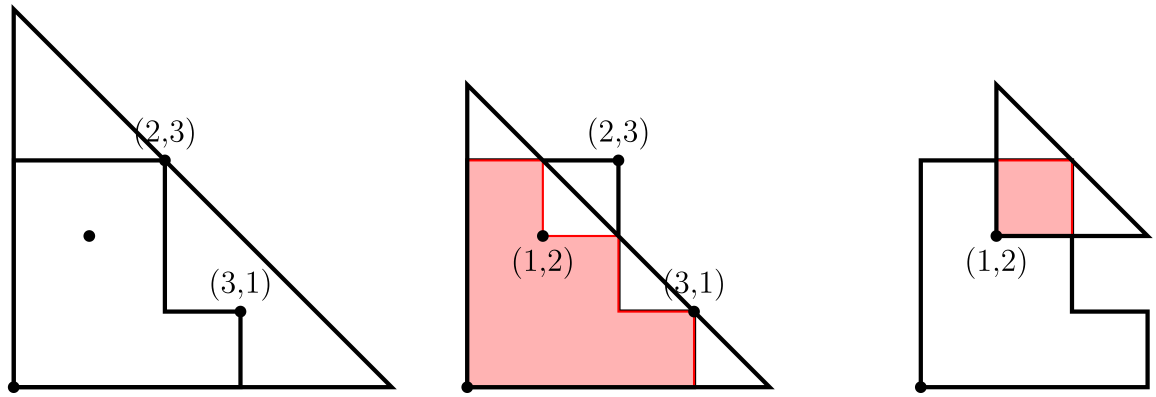

Figure 2.1: Covering of Tessellation of tile by multiple simplices

average of numbers greater than or equal to 1.5, and the density must be at least 1.5 in this case. ◻

Let us look at an example of this. Let the base lattice L be generated by $v_1 = (2, 1)$ and $v_2 = (-1, 3)$. Let $D = \{(1, 2)\}$. Now we note that the tile is broken up into a 3-staircase and a square. The density of this covering is given by $\delta_{L+D} = \frac{6}{7} * \frac{4^2}{2*6} + \frac{1}{7} * \frac{2^2}{2}$. So we have that the density of this multilattice covering is $\delta_{L+D} = \frac{10}{7}$. This is illustrated in Figure 2.1.

Notice that the 3-staircase prototile is covered with a simplex that is $\frac{8}{7}$ the area of the staircase. The rectangular prototile is covered by a simplex that is 2 times the area of the rectangle. In general, the lowest possible ratio we can have of a simplex to a corresponding rectangular prototile is 2. But we have shown that without this occurrence, we cannot obtain a covering density of less than 1.5. By choosing a displacement such that the rectangular prototile is a small proportion of the area of the tile, the multilattice covering density may be reduced. We now approach the problem of how to place a single displacement in relation to the tile.

Theorem 2.10. *The multilattice with a single displacement that results in the lowest possible covering density must have the density at a point $(a - h, d - h)$, where (a, d) is the corner of T which defines the Manhattan diameter of T, and $h > 0$.*

Proof. Let (a, d) be the point with the largest Manhattan diameter from the origin in our tile. We must place the displacement at $(a, d) - (h_1, h_2)$ by Theorem 2.9. Let λ_1 be the Manhattan diameter of the outermost corner. Let λ_2 be the Manhattan diameter of the next outermost corner, which is the point (c, b). The diameter of the simplex used in the lattice covering is λ_1. Under the tessellation of the tile based on the displacement, we will get a rectangular prototile and a 3-staircase prototile. The Manhattan diameter of the 3-staircase prototile is λ_1 decreased by $\min\{h_1, h_2, (\lambda_1 - \lambda_2)\}$. Our three corners for our 3-staircase will be the points $(c, b), (a - h_1, d), (a, d - h_2)$. The one with the greatest Manhattan diameter from the origin will determine the diameter of the simplex we will use to cover the 3-staircase. Thus the diameter of the base simplex has a Manhattan diameter that is $\min\{h_1, h_2, \lambda_1 - \lambda_2\}$ smaller than the Manhattan diameter used in the lattice covering. The rectangular prototile will have diameter $h_1 + h_2$. So the area of the simplex which covers the rectangular prototile must be $\frac{(h_1+h_2)^2}{2}$. This area increases with a larger choice of either h_1 or h_2. We also note that the area of the base simplex will be $\frac{(\lambda_1 - \min\{h_1, h_2, \lambda_1 - \lambda_2\})^2}{2}$. If $h_1 > \lambda_1 - \lambda_2$ and $h_2 > \lambda_1 - \lambda_2$, then we note that the area of the base simplex will be $\frac{\lambda_2^2}{2}$. In this case, we note that by reducing h_i, we only vary the area of the displacement simplex. When $h_i = \lambda_1 - \lambda_2$, the area of the base simplex remains the same, but the area of the simplex covering the rectangular prototile is reduced. So we should choose our displacement such that $h_1 \leq (\lambda_1 - \lambda_2)$ and $h_2 \leq (\lambda_1 - \lambda_2)$. Now we worry about what we may choose for h_1 and h_2 within these constraints. Let us assume, without loss of generality, that $h_1 \geq h_2$. Let the area of T be denoted by A_T. Then we note that the area of the base simplex will be $\frac{(\lambda_1 - h_2)^2}{2}$. Let this be denoted by S_B. If we decrease h_1 as much as possible, this will not affect S_B, since S_B only depends on the smaller value of h_1 and h_2. Now we note that the area of the displacement simplex will be $\frac{(h_1+h_2)^2}{2}$. This will be decreased if we decrease the value of h_1. Note that the density of the multilattice covering now becomes

$$\delta = \frac{h_1^2 + 2h_1h_2 + h_2^2 + \lambda_1^2 - 2\lambda_1 h_2 + h_2^2}{2A_T} = \frac{h_1^2 + 2h_1h_2 + 2h_2^2 + \lambda_1^2 - 2\lambda_1 h_2}{2A_T}.$$

The values of λ_1 and A_T are both fixed for a given tile. Now if we fix h_2, we note that δ will be determined by h_1. Taking the derivative of this function with respect to h_1 under these assumptions we get

$$\delta' = \frac{2h_1 + 2h_2}{2A_T} = \frac{h_1 + h_2}{A_T}$$

and we note that δ' gives positive values for any positive value of h_1. Thus, the smallest possible value of δ will occur when the smallest value possible for h_1 is chosen. Based on the constraints we have set, this gives us that $h_1 = h_2$ will be the best choice for where to place our displacement. So our displacement must be $(a - h_1, d - h_1)$. □

2.4 Square tile problem

Now we will solve where our displacement should go with the simplest case for our tile. Let us take a tile that is a square.

Theorem 2.11. *Let L be a lattice that results in a square tile from the subtraction construction. The lowest density possible for a multilattice $L + D$ with the $|D| = 2$ is 1.6.*

Proof. Let L be a lattice that generates a square tile under the subtraction construction. Let us scale this lattice so that our tile has an area of 1. We note that the lattice covering of $\mathbb{R}^2$ by L must result in a density of 2. If we were to add a single displacement, we may achieve a better density. By Theorem 2.10 we note that this displacement will be located at $(1-h, 1-h)$. Let S_B be the area of the base simplex and S_D be the area of the displacement simplex. We then have that

$$S_B = \frac{(2-h)^2}{2} = \frac{4 - 4h + h^2}{2}$$
$$S_D = \frac{(h+h)^2}{2} = 2h^2.$$

With this information, we again note that the density function is the ratio of the total area of the simplices divided by the area of the tile, and so we get the density function of

$$\delta = \frac{S_B + S_D}{1} = \frac{4 - 4h + 5h^2}{2}$$

for the multilattice with one displacement. Now to find the minimum we find the derivative of this function with respect to h.

$$\delta' = \frac{-4 + 10h}{2} = -2 + 5h.$$

We set this equal to zero, and we find that $\delta' = 0$ when $h = \frac{2}{5}$. We note that for any value of $h < \frac{2}{5}$, we get a negative value for δ', and for any $h > \frac{2}{5}$ we get a positive value for δ', confirming this value as a minimum. So we place the displacement at $\left(\frac{3}{5}, \frac{3}{5}\right)$. When we do this we get a density of

$$\delta = \frac{4 - 4\left(\frac{2}{5}\right) + 5\left(\frac{2}{5}\right)^2}{2} = \frac{8}{5}. \qquad \square$$

This density is lower than the density of 2 we originally had. We also note that by Theorem 2.8 and Theorem 2.10 this is the best multilattice covering of this tile type with one displacement. We also note that this density is greater 1.5, so it will not be the best multilattice covering with one displacement.

2.5 Best improvement for rectangular tile

We now see what the best density is for a multilattice with one displacement and a rectangular shaped tile for the base lattice L. We will first examine the case where the prototile placed at our displacement is a rectangle, and the prototile at the origin is a 2-staircase.

Theorem 2.12. *Let L be a lattice that gives a rectangular prototile under the subtraction construction. Any multilattice $L + D$ with $|D| = 2$ and a rectangular prototile will have density of at least 1.6.*

Proof. To begin, we scale the lattice in such a way that the Manhattan diameter of the base tile is 1. Let the tile have dimensions $x \times y$. This gives us that the area of the simplex used in the lattice covering will be $\frac{1}{2}$. Notice that if we place the displacement (h, h) behind the outermost corner, the Manhattan diameter of the part of the tile covered by the base simplex will be $1-h$ and the Manhattan diameter of the displacement will be $2h$. So we now have a density of

$$\delta = \frac{\frac{(1-h)^2}{2} + \frac{(2h)^2}{2}}{xy} = \frac{1-2h+5h^2}{2xy}$$

where x and y are both positive. We now note that since $x+y=1$, $y=1-x$. We also have that $x, y \in (0,1)$. We also note that if we find the optimal value of h in terms of x, we must minimize

$$\delta = \frac{1-2h+5h^2}{2x-2x^2}.$$

To do this, we find the derivative with respect to h, and we get

$$\delta' = \frac{-2+10h}{2x-2x^2}$$

and we set this equal to zero to get the critical numbers.

$$0 = \frac{-2+10h}{2x-2x^2}$$
$$0 = -2+10h$$
$$h = \frac{1}{5}.$$

We now check to see if this is a minimum. We notice that since $x \in (0,1)$, $x > x^2$, and so we note that $2x-2x^2 > 0$. So we need only check the sign of the numerator. We note that for $h < \frac{1}{5}$, $\delta' < 0$. We also note that for any $h > \frac{1}{5}$, we have that $\delta' > 0$. So the value of $h = \frac{1}{5}$ gives us a minimum for δ. Notice that by our choice of scaling, we still have that $y = 1-x$. We also note the displacement will be $\left(\frac{1}{5}, \frac{1}{5}\right)$ behind the outermost corner. Thus the base simplex will have an area of $\frac{8}{25}$ and the displacement simplex will have an area of $\frac{2}{25}$. Thus the density of the multilattice simplex covering will be given by

$$\delta = \frac{\frac{8}{25} + \frac{2}{25}}{2x-2x^2} = \frac{1}{5x-5x^2}.$$

To minimize this we use the same process. We find the derivative with respect to x, and we get

$$\delta' = \frac{-(5-10x)}{(5x-5x^2)^2} = \frac{2x-1}{5x^2-10x^3+5x^4}.$$

We note that the denominator factors into $5x^2(x-1)^2$, and since 1 and 0 are not possible values for x, the only critical number we may get is when δ' is equal to 0. This happens when we set $x = \frac{1}{2}$. Now we note that if $x < \frac{1}{2}$, then $\delta' < 0$, and if $x > \frac{1}{2}$ then $\delta' > 0$, so this is the minimum density we may achieve. When $x = \frac{1}{2}$, $y = \frac{1}{2}$ as well. So the best possible multilattice covering with a base lattice that gives a rectangular tile with one displacement will be a scaling of the multilattice $\mathbb{Z}^2 + (\frac{3}{10}, \frac{3}{10})$. If we scale this by a factor of 2, we get the example from the previous section, so the density is $\frac{8}{5}$. □

While a displacement of this type will result in a covering density of 1.6, it should be noted that this is not the minimum covering density of any multilattice $L + D$ with a single displacement and T a rectangular tile.

Theorem 2.13. *Let L be a lattice that results in a rectangular tile under the subtraction construction. Let $|D| = 2$. The minimum multilattice covering density of any multilattice of the form $L + D$ is 1.5.*

Proof. Let L be any lattice that results in a rectangular tile under the subtraction construction. We note that for any lattice of this type we may select a lattice point $(0, c)$ or $(c, 0)$ to be a generator of the lattice. This is because under any tiling with a rectangle, each corner is identified with at least one other corner. As T has a corner on the origin, at least one other must be identified with it. Without loss of generality, let $(0, c)$ be a generator of L. Now let (x, y) denote a displacement. If the prototile at (x, y) is a rectangle, we may only achieve a density of 1.6. So we now look at the case where the simplex placed at $(x, y) + g$ for some g in L covers the point (x, d), where d is the height of the rectangle. If we translate this region covered by the simplex at $(x, y) + g$ by $-g$, then we note that we have a tile that is the union of 2 2-staircases. We have already shown that 1.5 is a lower bound for multilattice covering density in this case. Now take the lattice generated by $(0, 3)$ and $(2, -1)$. This has covolume of 6. Now choose $(1,1)$ as our displacement. Place a simplex of scale 3 at this point. The rectangle $(1,1)$ to $(2,3)$ lies in the simplex placed at $(1,1)$. We also note that the rectangle $(0,2)$ to $(1,3)$ lies in the simplex placed at $(1,1)-(2,-1)=(-1,2)$. So the region left uncovered by the displacement simplex is the 2-staircase placed at the origin with corners $(1,2)$ and $(2,1)$. Place another simplex of scale 3 at the origin. This generates a cover, with 2 simplices each of area 4.5. Thus $\delta = \frac{4.5+4.5}{6} = \frac{9}{6} = 1.5$ for the multilattice covering. □

These theorems show us that the lowest covering density of a multilattice with a single displacement must arise from a tile that is a 2-staircase.

2.6 L-shaped tile optimization with 1 displacement

Now we find the best multitlattice covering density with one displacement for a lattice that gives us an L-shaped tile with the subtraction construction.

Theorem 2.14. *Let T be any tile that is a 2-staircase. Let $L+D$ be a mulilattice such that $|D|=2$. The greatest lower bound for the covering density of $L+D$ is $5-\sqrt{13}$ (about 1.39445).*

Proof. Let T be any tile that is a 2-staircase. Let λ_1 and λ_2 be the Manhattan diameter of the corners of T, such that $\lambda_1 \geq \lambda_2$. We notice by Theorem 2.10 we will get a square prototile covered by our displacement. The best multilattice covering will be achieved when $\lambda_1 - h \geq \lambda_2$, as we have seen in the proof of Theorem 2.10

We first check the case where $\lambda_1 - h > \lambda_2$. This results in the portion of the tile covered by the base simplex having two corners with Manhattan diameter $\lambda_1 - h$, and a third with Manhattan diameter λ_2. However, notice that if we were to deform the 2-staircase in such a way that λ_1 remained constant but λ_2 increased, we would have an increased area of T, while the Manhattan diameter of T remains the same. So the ratio of the area of the simplices used to cover T to the area of T would decrease. This is illustrated in Figure 2.2. So we may obtain a lower density simply by increasing λ_2 until $\lambda_2 = \lambda_1 - h$.

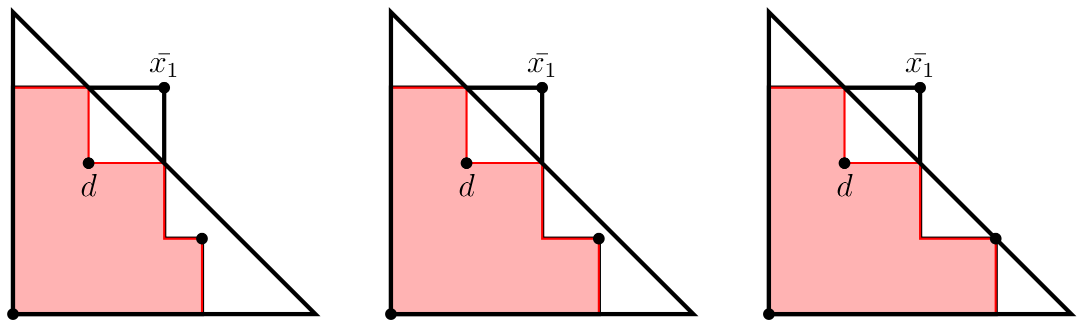

Figure 2.2: Increasing λ_2 for an L-shaped tile

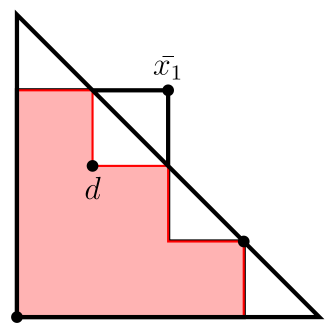

Figure 2.3: 3-step staircase and simplex

We now have that it must be the case that $\lambda_1 - h = \lambda_2$. Let (a, d) be the corner of T with Manhattan diameter λ_1 Then when we place one simplex at $(a, d) - (h, h)$, the portion of T covered by the remaining simplex is a three-step staircase with all three corners of equal Manhattan diameter. We will also scale the lattice such that the base simplex will have a hypotenuse of length 1. We notice that the base simplex will cover an area of the tile as is shown in Figure 2.3.

We may minimize the overlap of this simplex by minimizing the area of the four right triangles whose hypotenuses sum to 1. We note that one of these will be defined by the value of h, but the other three are dependent on the coordinates of the corners of T. For this purpose we will determine what the relation between the three triangles dependent on T should be.

To do this, set $h = 0$. Then we note that if the triangles have hypotenuses of length $\sqrt{2}a$, $\sqrt{2}b$, and $\sqrt{2}c$, we have that $\sqrt{2}a+\sqrt{2}b+\sqrt{2}c = 1$. So we have that $\sqrt{2}c = 1-\sqrt{2}a-\sqrt{2}b$.

We now note that the total area of these three triangles is

$$A = \frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2} = \frac{a^2 + b^2 + \left(\frac{1}{\sqrt{2}} - a - b\right)^2}{2} = \frac{4a^2 + 4ab - 2\sqrt{2}a + 4b^2 - 2\sqrt{2}b + 1}{4}.$$

We now take the partial derivatives with respect to a and b and we get

$$A_a = \frac{4a + 2b - \sqrt{2}}{2}$$
$$A_b = \frac{2a + 4b - \sqrt{2}}{2}.$$

Finding where these are both zero we get that

$$0 = \frac{4a + 2b - \sqrt{2}}{2}$$
$$0 = \frac{2a + 4b - \sqrt{2}}{2}.$$

We solve for b in the first and we get

$$b = \frac{\sqrt{2} - 4a}{2}.$$

Now we substitute this into the second equation and we get

$$0 = \frac{2a + 4\left(\frac{\sqrt{2}-4a}{2}\right) - \sqrt{2}}{2} = \frac{\sqrt{2} - 6a}{2}.$$

This gives that $a = \frac{\sqrt{2}}{6}$, and that $b = \frac{\sqrt{2}}{6}$. We also have from before that $c = \frac{1}{\sqrt{2}} - a - b$ and so we have that $c = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{3} = \frac{\sqrt{2}}{6}$. Thus we have that $a = b = c$. We now check the second partial derivatives and we get

$$A_{aa} = 2$$

$$A_{bb} = 2$$

$$A_{ab} = 1.$$

Now we have that $A_{aa}A_{bb} - A_{ab}^2 = 3 > 0$ and $A_{aa} > 0$ so we have a minimum at this point. Thus the smallest possible area of these three triangles occurs when they all have the same length of hypotenuse. Notice that the choice of 1 for the sum of the hypotenuse does not affect the relation, so $a = b = c$. Thus, we must choose a base lattice in such a way that the three triangles determined by the base lattice are congruent.

We now turn to determining what value of h will give us the lowest possible density. We now note that the Manhattan diameter of the base simplex will still be $\lambda_1 - h$ and the Manhattan diameter of the displacement simplex will be $2h$. We again scale so that the hypotenuse of the base simplex will be 1. Now we determine what the area of the tile will be. We note that if we set the three triangles from the base lattice congruent, and then have a non-zero value for h, the hypotenuse of the base simplex is $3\sqrt{2}b+\sqrt{2}h$. Note that we scale so that this is 1, and we get $b = \frac{1-\sqrt{2}h}{3\sqrt{2}}$. Now we may express the area of the tile in terms of h and b. Notice that $\sqrt{2}b$ is the length of the hypotenuse of any of the three triangles that are of equal volume, and that $\sqrt{2}h$ is the length of the hypotenuse of the triangle that overlaps with the displacement simplex if we scale such that the base simplex has a hypotenuse of length 1. We now have that the area of the tile will be

$$A_T = (h + 2b)(h + b) + b^2 = h^2 + 3bh + 3b^2$$

as the tile may be divided into two rectangles. One a square of side length b, and the other a rectangle with one side of length $h + 2b$ and the other of length $h + b$. Since the hypotenuse of the base simplex is 1, we note that the area of the base simplex will be fixed at a value of $\frac{1}{4}$. The Manhattan diameter of the displacement simplex is $2h$, so the displacement simplex will have an area of $2h^2$. Thus we have for our density of the multitlattice covering

$$\delta = \frac{\frac{1}{4} + 2h^2}{h^2 + 3bh + 3b^2} = \frac{24h^2 + 3}{4h^2 + 2\sqrt{2}h + 2}.$$

Now we note that once again we must find the minimum value of this function. So we take the derivative with respect to h and we get

$$\delta' = \frac{(4h^2 + 2\sqrt{2}h + 2)(48h) - (24h^2 + 3)(8h + 2\sqrt{2})}{(4h^2 + 2\sqrt{2}h + 2)^2}$$
$$= \frac{24\sqrt{2}h^2 + 36h - 3\sqrt{2}}{8h^4 + 8\sqrt{2}h^3 + 12h^2 + 4\sqrt{2}h + 2}.$$

To find the critical numbers, we check the denominator, and note that it may be expressed as $2(2h^2+\sqrt{2}h+1)^2$. So we must find where this polynomial is equal to zero. By the quadratic formula, we have that

$$h = \frac{-\sqrt{2} \pm \sqrt{2-8}}{4} = \frac{-\sqrt{2} \pm \sqrt{-6}}{4}$$

so there is no real value of h such that $2(2h^2 + \sqrt{2}h + 1)^2 = 0$. So we now check where the numerator is zero. If

$$0 = 24\sqrt{2}h^2 + 36h - 3\sqrt{2}$$

then we know

$$h = \frac{-36 \pm \sqrt{1296 + 576}}{48\sqrt{2}} = \frac{-3 \pm \sqrt{13}}{4\sqrt{2}}.$$

We note that only the positive value will lie in our domain, so the only critical number we must worry about is when $h = \frac{-3+\sqrt{13}}{4\sqrt{2}}$. We note that $\delta' < 0$ if $h < \frac{-3\pm\sqrt{13}}{4\sqrt{2}}$ and that $\delta' > 0$ if $h > \frac{-3\pm\sqrt{13}}{4\sqrt{2}}$, so this is a minimum. Notice that this is the minimum for all values of h such that $h \in \left[0, \frac{1}{\sqrt{2}}\right]$, which are all possible values of h if we scale the base simplex to have a hypotenuse of 1. Thus, this will be the absolute minimum density for a multilattice with one displacement and an L-shaped base tile. Plugging this value in to our density function, we get

$$\delta = \frac{24\left(\frac{-3\pm\sqrt{13}}{4\sqrt{2}}\right)^2 + 3}{4\left(\frac{-3\pm\sqrt{13}}{4\sqrt{2}}\right)^2 + 2\sqrt{2}\left(\frac{-3\pm\sqrt{13}}{4\sqrt{2}}\right) + 2} = 5 - \sqrt{13}.$$

This about 1.39445, which is lower than the value of $\frac{8}{5}$. So the minimum possible density of a multilattice simplex covering with one displacement is $5 - \sqrt{13}$. $\square$

2.7 Obtaining the density

Now that we have a lower bound, we must see if it is possible to obtain this value. Let us take the lattice that is generated by the two vectors

$$v_1 = \left(\frac{-2\sqrt{2}+2\sqrt{26}}{24}, \frac{7\sqrt{2}-\sqrt{26}}{24}\right)$$
$$v_2 = \left(\frac{\sqrt{26}-7\sqrt{2}}{24}, \frac{5\sqrt{2}+\sqrt{26}}{24}\right).$$

We check that these two vectors are not colinear, and so we check the angle θ between the two is nonzero.

$$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\|\,\|v_2\|} = \frac{\frac{31-7\sqrt{13}}{144}}{\frac{\sqrt{3236-668\sqrt{13}}}{144}} = \frac{31-7\sqrt{13}}{\sqrt{3236-668\sqrt{13}}} \neq 1.$$

So we do get a rank 2 lattice from these two vectors. Next, we place these as the columns in a matrix, and find the determinant. This gives us the volume of the tile is

$$\begin{vmatrix} \frac{\sqrt{26}-\sqrt{2}}{12} & \frac{\sqrt{26}-7\sqrt{2}}{24} \\ \frac{7\sqrt{2}-\sqrt{26}}{24} & \frac{\sqrt{26}+5\sqrt{2}}{24} \end{vmatrix} = \frac{13-\sqrt{13}}{48}.$$

We now check the resulting tile to see what the Manhattan diameter will be. Using the subtraction construction, we find the lattice points that will be the blockers for our tile. We note that v_1 will be in the first quadrant, as both the x and y coordinate are positive, and so is a potential blocker. Next we note that v_2 is a potential blocker as the sum of coordinates is positive. Finally, let us take $v_1 - v_2$. The resulting point is $(\frac{\sqrt{26}+5\sqrt{2}}{24}, \frac{2\sqrt{2}-2\sqrt{26}}{24})$, which also has a positive sum of its coordinates. These three are therefore potential blockers for our tile. Now we note that $(F \setminus (F+v_2)) \setminus (F+(v_1-v_2)) = F''$, we have that the x coordinate for any point in our tile may be no more than $\frac{\sqrt{26}+5\sqrt{2}}{24}$ and the y coordinate for any point in our tile may be no more than $\frac{\sqrt{26}+5\sqrt{2}}{24}$. Now we take $F'' \setminus (F+v_1)$ from the square we have, and we get an L-shape with outer corners at the points $(\frac{\sqrt{26}+5\sqrt{2}}{24}, \frac{7\sqrt{2}-\sqrt{26}}{24})$

and $(\frac{-2\sqrt{2}+2\sqrt{26}}{24}, \frac{\sqrt{26}+5\sqrt{2}}{24})$. The area of this shape T will be

$$\begin{aligned} A_T &= \left(\frac{-2\sqrt{2}+2\sqrt{26}}{24}\right)\left(\frac{\sqrt{26}+5\sqrt{2}}{24}\right) + \left(\frac{-\sqrt{26}+7\sqrt{2}}{24}\right)\left(\frac{7\sqrt{2}-\sqrt{26}}{24}\right) \\ &= \frac{13-\sqrt{13}}{48}. \end{aligned}$$

This is the covolume of the lattice, so we know that the shape T is our tile for this lattice. Now we must determine where the displacement should go to optimize the density. Note that $\lambda_1 = \frac{3\sqrt{26}+3\sqrt{2}}{24}$ and $\lambda_2 = \frac{12\sqrt{2}}{24}$. We will place the displacement (h, h) behind the point $(\frac{-2\sqrt{2}+2\sqrt{26}}{24}, \frac{\sqrt{26}+5\sqrt{2}}{24})$. We impose the same restrictions on h as before. So the density will be

$$\begin{aligned} \delta &= \frac{(\frac{3\sqrt{26}+3\sqrt{2}}{24} - h)^2 + (2h)^2}{\frac{13-\sqrt{13}}{48}} \\ &= \frac{(20\sqrt{13}+260)h^2 + (-14\sqrt{26}-26\sqrt{2})h + 5\sqrt{13}+26}{26}. \end{aligned}$$

The derivative of this function with respect to h is

$$\delta' = \frac{(20\sqrt{13}+260)h - 7\sqrt{26} - 13\sqrt{2}}{13}.$$

This is 0 when $h = \frac{\sqrt{26}+\sqrt{2}}{40}$. However, we note that

$$\lambda_1 - \lambda_2 = \frac{3\sqrt{26}-9\sqrt{2}}{24} < \frac{\sqrt{26}+\sqrt{2}}{40}$$

so there are no critical points in our domain that we may choose for our value of h. We also note that for all values of h that we may choose, $\delta' < 0$, and so to minimize the density, we must choose the largest possible value for h. This is $\lambda_1 - \lambda_2$, and so we have that $h = \left(\frac{\sqrt{26}-3\sqrt{2}}{8}\right)$. When we plug this value into our density function, we get back the value of

$$\delta = \frac{(20\sqrt{13}+260)\left(\frac{\sqrt{26}-3\sqrt{2}}{8}\right)^2 + (-14\sqrt{26}-26\sqrt{2})\left(\frac{\sqrt{26}-3\sqrt{2}}{8}\right) + 5\sqrt{13}+26}{26} = 5-\sqrt{13}.$$

So the value is obtainable by a multilattice, though not an integral multilattice.

Chapter 3. Multiple Displacements in Two Dimensions

Now that we have addressed the problem with one nontrivial displacement, we see that we may improve on the covering density of a lattice by adding a set of displacements. We now explore what happens when we add multiple displacements, and how each may affect the total covering density.

3.1 Three displacements in two dimensions

To see how we may improve, we first look at the problem when we allow for two displacements. We will limit ourselves to connected prototiles of T for the time being. We note that there are two cases which may result in the optimal covering density for a multilattice with two displacements.

Theorem 3.1. *The minimum multilattice covering density of a multilattice with 2 displacements covering connected prototiles is* $\frac{4}{3}$

Proof. We break the problem into two cases. The first case is that we start with an L-shaped tile, and use both nontrivial displacements to cover a single square section of the base tile, as in Figure 3.1. We note by Theorem 2.11 that the most efficient covering of a square region by two simplices will result in a density of 1.6. For this reason, we may also consider the

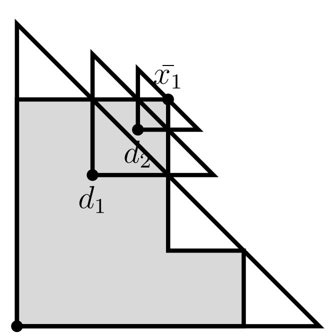

Figure 3.1: Two displacements cover a square region of our tile

problem as improving a lattice covering with one nontrivial displacement, but rather than having the area of the displacement simplex be equal to $2h^2$, or twice the area of the square prototile, we set it equal to $\frac{8h^2}{5}$. So now we note that with these two displacements, we may once again approach this as we did before. We scale the lattice so that the hypotenuse of the base simplex will be 1. We choose the lattice so that the three regions of overlap not covered by the displacement simplex will be equal. We define b in the same way as before, so that $b = \frac{1-\sqrt{2}h}{3\sqrt{2}}$. Now we have that

$$A_T = (h + 2b)(h + b) + b^2 = h^2 + 3bh + 3b^2$$

and we are now able to find our density function. This will be

$$\delta = \frac{\frac{1}{4} + \frac{8h^2}{5}}{\frac{2h^2+\sqrt{2}h+1}{6}} = \frac{96h^2 + 15}{20h^2 + 10\sqrt{2}h + 10}$$

and we get

$$\delta' = \frac{3\left(32\sqrt{2}h^2 + 44h - 5\sqrt{2}\right)}{10\left(2h^2 + \sqrt{2}h + 1\right)^2}$$

for the derivative with respect to h. The quadratic factor in the denominator has no real roots, and so we need only worry about when

$$0 = 96\sqrt{2}h^2 + 132h - 15\sqrt{2}.$$

This occurs when

$$h = \frac{-132 \pm \sqrt{28944}}{192\sqrt{2}} = \frac{-11 \pm \sqrt{201}}{16\sqrt{2}}.$$

Only the positive option is possible, so we check this critical number. We note that $\delta' < 0$ when $h < \frac{-11+\sqrt{201}}{16\sqrt{2}}$ and $\delta' > 0$ when $h > \frac{-11+\sqrt{201}}{16\sqrt{2}}$, so this is a minimum. Now we check the

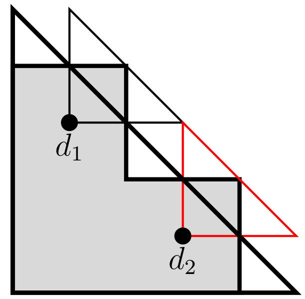

Figure 3.2: 2 displacements each cover a corner

value of δ when $h = \frac{-11+\sqrt{201}}{16\sqrt{2}}$.

$$\delta = \frac{96\left(\frac{-11+\sqrt{201}}{16\sqrt{2}}\right)^2 + 15}{20\left(\frac{-11+\sqrt{201}}{16\sqrt{2}}\right)^2 + 10\sqrt{2}\left(\frac{-11+\sqrt{201}}{16\sqrt{2}}\right) + 10} = \frac{21-\sqrt{201}}{5}.$$

This is about 1.36451, a slight improvement on the density of a multilattice with one displacement.

We now examine the case where we have two distinct displacements cover both corners of an 2-staircase. We note that this will give a 4-staircase that is covered by the base simplex, and two square regions covered by the two displacement simplices, as in Figure 3.2. Once more, we will scale the lattice so that the base simplex will be covered by a simplex with hypotenuse of length 1. Now note that we have two displacement simplices, one will have side length h_1 and the other with side length h_2. We also note that the displacement simplices must have areas $2h_1^2$ and $2h_2^2$. Now if we look at the area of the tile we note that we will get

$$A_T = (b+h_1)(h_1+h_2+2b) + (h_2+b)^2 = h_1^2 + h_1h_2 + 3h_1b + h_2^2 + 3h_2b + 3b^2.$$

We also note that $\frac{1}{3\sqrt{2}} - \frac{h_1}{3} - \frac{h_2}{3} = b$. With this we may now express the density function in terms of h_1 and h_2.

$$\delta = \frac{\frac{1}{4} + 2h_1^2 + 2h_2^2}{\frac{\sqrt{2}h_1+\sqrt{2}h_2-2h_1h_2+1+2h_1^2+2h_2^2}{6}} = \frac{3+24h_1^2+24h_2^2}{2\sqrt{2}h_1+2\sqrt{2}h_2-4h_1h_2+2+4h_1^2+4h_2^2}.$$

To minimize this function, we find the partial derivatives of this function. Let p be the value of $8h_2^4 + 8\sqrt{2}h_2^3 + 12h_2^2 + 4\sqrt{2}h_2 + 2$. The partial derivative with respect to h_1 is

$$\delta_{h_1} = \frac{-48h_1^2h_2 + 24\sqrt{2}h_1^2 + 48\sqrt{2}h_1h_2 + 36h_1 + 48h_2^3 - 24\sqrt{2}h_2^2 + 6h_2 - 3\sqrt{2}}{8h_1^4 - 16h_1^3h_2 + 8\sqrt{2}h_1^3 + 24h_1^2h_2^2 + 12h_1^2 - 16h_1h_2^3 + 4\sqrt{2}h_1 + p}.$$

We also find that the partial derivative with respect to h_2 is

$$\delta_{h_2} = \frac{-48h_2^2h_1 + 24\sqrt{2}h_2^2 + 48\sqrt{2}h_1h_2 + 36h_2 + 48h_1^3 - 24\sqrt{2}h_1^2 + 6h_1 - 3\sqrt{2}}{8h_2^4 - 16h_2^3h_1 + 8\sqrt{2}h_2^3 + 24h_1^2h_2^2 + 12h_2^2 - 16h_2h_1^3 + 4\sqrt{2}h_2 + p}.$$

Since both of these must be zero at a local minimum for the density, we may simplify the problem by noting that $\delta_{h_1} = 0 = \delta_{h_2}$ when $h_1 = h_2$. So we now work under the assumption that $h_1 = h_2$, and we will refer to it as h. Now we may express the density of the covering as

$$\delta = \frac{\frac{1}{4} + 4h^2}{\frac{2h^2+2\sqrt{2}h+1}{6}} = \frac{3 + 48h^2}{4h^2 + 4\sqrt{2}h + 2}$$

and find the derivative is

$$\delta' = \frac{48\sqrt{2}h^2 + 42h - 3\sqrt{2}}{\left(2h^2 + 2\sqrt{2}h + 1\right)^2}$$

which gives us that $\delta' = 0$ when $h = \frac{-14+\sqrt{324}}{32\sqrt{2}} = \frac{\sqrt{2}}{16}$. We again check that this gives us a minimum. When $h < \frac{\sqrt{2}}{16}$, we get $\delta' < 0$ and when $h > \frac{\sqrt{2}}{16}$, $\delta' > 0$. So this is a minimum. Now we note that when $h = \frac{\sqrt{2}}{16}$, we get a density of

$$\delta = \frac{3 + 48\left(\frac{\sqrt{2}}{16}\right)^2}{4\left(\frac{\sqrt{2}}{16}\right)^2 + 4\sqrt{2}\left(\frac{\sqrt{2}}{16}\right) + 2} = \frac{4}{3}$$

which is the lower of the two, so it must be that the minimum for two displacements covering connected prototiles is $\frac{4}{3}$. $\square$

So we once more decrease the covering density by adding one additional displacement. In this case we note that we decrease the density by $5 - \sqrt{13}$ (1.39445) to $\frac{4}{3}$ (1.33333) by adding a single displacement. Note that by adding one displacement to a lattice we decreased bound on the density from 1.5 to about 1.39445. By adding a second non-trivial displacement, we only decrease the bound on the density by about 0.061115. We also note that the multilattice that results in the density of $\frac{4}{3}$ may be scaled so that it is an integral multilattice.

3.2 Arbitrary displacement problem

Now that we have worked through some of the covering densities for a given number of displacements, we determine what the greatest lower bound on the density of any multilattice covering is. We will prove the following.

Theorem 3.2. *The greatest lower bound for the density of a multilattice covering of $\mathbb{R}^2$ with an arbitrary number of displacements is 1.*

Proof. We begin by noting that no multilattice will ever have a covering density of 1 while we only use dilations of the right regular simplex as our convex bodies. If 1 is a lower bound for any multilattice, we may work with a single base lattice. Let us choose the base lattice of $\mathbb{Z}^2$. If we can show that 1 is the greatest lower bound for a multilattice of the form $\mathbb{Z}^2 + D$, we are done. With $\mathbb{Z}^2$ as our base lattice, we get a square tile T of area 1. Choose $m = 2^k$, where k is a natural number. We now divide this tile into m^2 squares of equal area. Each square will have a side length of $\frac{1}{m}$ and an area of $\frac{1}{m^2}$. Let us place a simplex of diameter $\frac{m+1}{m}$ at the base lattice point. This simplex will cover $\frac{m^2+m}{2}$ of the squares. This simplex will cover these squares with a density of

$$\delta_1 = \frac{\left(\frac{m+1}{m}\right)^2 \frac{1}{2}}{\frac{m^2+m}{2m^2}} = \frac{m^2+2m+1}{m^2+m}.$$

The remaining $\frac{m^2-m}{2}$ squares lie above the diagonal of the tile T. We will do the first iteration of placing a simplex in this region. As m is even, then there are an odd number of squares directly above the diagonal. We place a simplex of Manhattan diameter $\frac{m+2}{2m}$ at $(\frac{1}{2}, \frac{1}{2})$. This will cover $\frac{\left(\frac{m}{2}\right)^2+\frac{m}{2}}{2} = \frac{m^2+2m}{8}$ squares of our tile. We will also get a density of

$$\delta_2 = \frac{\left(\frac{m+2}{2m}\right)^2\left(\frac{1}{2}\right)}{\frac{m^2+2m}{8m^2}} = \frac{m^2+4m+4}{m^2+2m}$$

for the newly covered section of our tile. Now we have three uncovered sections of our tile, but now we have $\frac{m^2-2m}{8}$ squares in each of them. We perform the second iteration of placing simplices by placing a simplex of scale $\frac{m+4}{4m}$ at the points $(\frac{1}{4},\frac{3}{4})$,$(\frac{3}{4},\frac{3}{4})$, and $(\frac{3}{4},\frac{1}{4})$. This will leave us with 9 connected regions that are not covered by any of these simplices. We repeat this process on the remaining regions. For the $i+1$th iteration, we will place 3^i simplices of scale $\frac{m+2^i}{2^i m}$ in our uncovered region. Let $\{d_{\{i,1\}}, d_{\{i,2\}}, \ldots, d_{\{i,3^{i-1}\}}\}$ be the points where we placed simplices in the ith iteration. We place simplices at the points

$$\{d_{i,j} + \left(\frac{n_1'}{2^i}, \frac{n_2'}{2^i}\right) \mid n_1', n_2' = \pm 1, n_1' + n_2' \geq 0\}$$

in the $i+1$th iteration. we continue this process until we place simplices of scale $\frac{2}{m}$. This will completely cover our tile T. The density of this covering will be

$$\delta = \left(\frac{m^2+m}{2m^2}\right)\left(\frac{m^2+2m+1}{m^2+m}\right) + \sum_{i=0}^{k-1} 3^i \left(\frac{m^2+2^{i+1}m}{2^{2i+3}m^2}\right)\left(\frac{m^2+2^{i+2}m+2^{2i+2}}{m^2+2^{i+1}m}\right)$$

and will have $1+\sum_{i=0}^{k-1} 3^i$ simplices in the covering. As $m = 2^k$, we rewrite this as

$$\left(\frac{2^{2k}+2^{k+1}+1}{2^{2k+1}}\right) + \sum_{i=0}^{k-1} 3^i \left(\frac{2^{2k}+2^{i+2}2^k+2^{2i+2}}{2^{2i+3}2^{2k}}\right).$$

We take the limit

$$\lim_{k\to\infty}\left[\left(\frac{2^{2k}+2^{k+1}+1}{2^{2k+1}}\right) + \sum_{i=0}^{k-1} 3^i \left(\frac{2^{2k}+2^{i+2}2^k+2^{2i+2}}{2^{2i+3}2^{2k}}\right)\right] = 1.$$

So 1 must be the greatest lower bound on the density of any multilattice. □

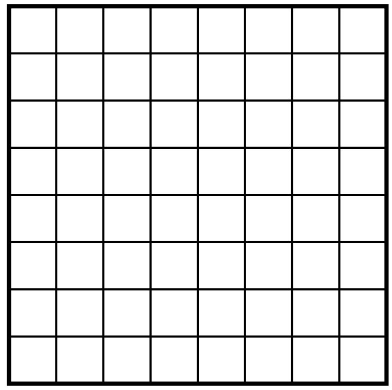

Figure 3.3: square tile divided into 64 squares

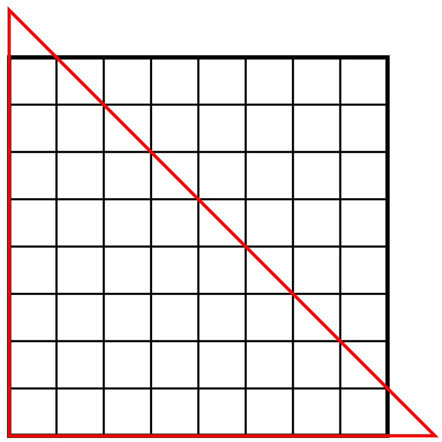

Figure 3.4: First simplex added

To make the process clear, we will illustrate this process when $m = 8$. To begin we take the tile of $\mathbb{Z}^2$ and divide it into 64 squares of equal area as in Figure 3.3. Each square will have side length $\frac{1}{8}$. Now we will place a simplex of scale $\frac{8+1}{8} = \frac{9}{8}$ at the origin. The region this covers is illustrated in Figure 3.4. Notice that the area not covered by this first simplex is congruent to a reflection of a 7-staircase about the origin. We now add our second simplex at the point $(0.5, 0.5)$ as in Figure 3.5. This simplex will have scale $\frac{(8)+2}{2(8)} = \frac{5}{8}$. Now we are left with three equal regions of squares that are not contained in one of these simplices. For each of these regions, we will place a simplex of scale $\frac{3}{8}$. We place one simplex at the point $(\frac{1}{4}, \frac{3}{4})$, one at $(\frac{3}{4}, \frac{3}{4})$, and one at $(\frac{3}{4}, \frac{1}{4})$, as in Figure 3.6. We are now left with 9 squares that are not contained in any of these simplices on our tile. We now cover each of these regions with a simplex of scale $\frac{1}{4}$. We note that T is now completely covered. This is shown in

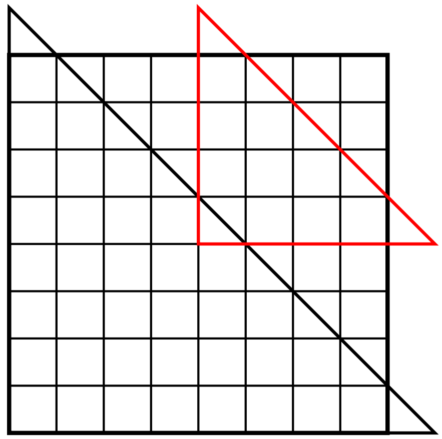

Figure 3.5: Second simplex added

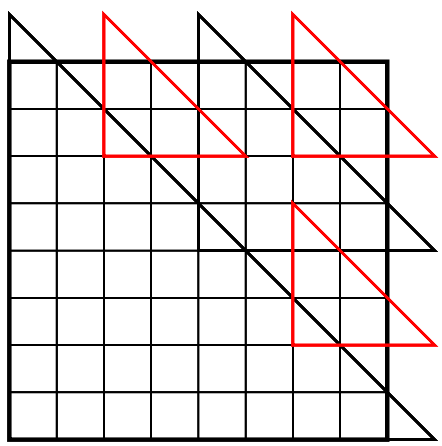

Figure 3.6: Adding simplices for first iteration of remaining space

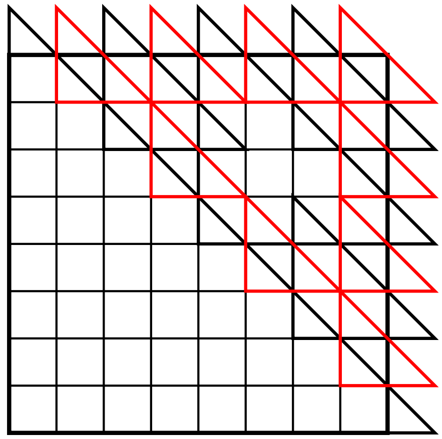

Figure 3.7: Adding simplices for 2nd iteration of remaining space

Figure 3.7. This gives us a covering density of

$$\delta = \frac{(8)^2 + 2(8) + 1}{2(8)^2} + \sum_{i=0}^{3} 3^i \left(\frac{8^2 + 2^{i+2}(8) + 2^{2i+2}}{2^{2i+3}(8)^2} \right) = \frac{169}{128}.$$

This density is 1.3203125. As we increase the value of m, we note that this will also decrease the density we get for our covering.

One more area of interest is where we might place a single displacement to have a multilattice covering density under a given value. For example, we know that the best density for a multilattice with one displacement is $5-\sqrt{13}$. Given the same base lattice L, where may we place the displacement and still get a density lower than 1.45? We look at some examples.

4.1 Square tile

Let us take the lattice $\mathbb{Z}^2$. Suppose that we want to add one displacment to get a multilattice that has a density between 2 and $\frac{8}{5}$. We will show the following.

Proposition 4.1. *Let L be a lattice that results in a square tile T of area 1 under the subtraction construction. Let c be a number such that $1.6 \leq c < \delta_L = 2$ for the lattice L. Let $D = \{0, (x, y)\}$. Then for any choice of (x, y) that lies inside the curve defined by*

$$c = \begin{cases} d_1^2 + d_1 d_2 - d_1 + \frac{1}{2} d_2^2 - 2d_2 + \frac{5}{2} & \text{if } d_1 \geq d_2 \\ d_2^2 + d_1 d_2 - d_2 + \frac{1}{2} d_1^2 - 2d_1 + \frac{5}{2} & \text{if } d_2 > d_2 \end{cases}$$

the multilattice covering density of $L + \{0, (x, y)\}$ will be at most c.

Proof. Let the displacement be $d = (d_1, d_2)$. Then we note that the Manhattan diameter of the displacement simplex will be $2 - d_1 - d_2$. Notice that this is in terms of the coordinates of our displacement. Previously we have worked with the distance from the displacement to the outermost corner. We also note that the Manhattan diameter of the base simplex will be $1 + \max\{d_1, d_2\}$. The area of the tile will be 1, so our density for our multilattice will be given by

$$\delta = \frac{(1 + \max\{d_1, d_2\})^2 + (2 - d_1 - d_2)^2}{2}.$$

If it is the case that $d_1 \geq d_2$, then we have that

$$\delta = \frac{(1+d_1)^2 + (2-d_1-d_2)^2}{2} = d_1^2 + d_1 d_2 - d_1 + \frac{1}{2}d_2^2 - 2d_2 + \frac{5}{2}.$$

If it is the case that $d_2 > d_1$, then the density becomes

$$\delta = \frac{(1+d_2)^2 + (2-d_1-d_2)^2}{2} = d_2^2 + d_1 d_2 - d_2 + \frac{1}{2}d_1^2 - 2d_1 + \frac{5}{2}.$$

To see where these two functions agree, we set them equal to each other and we get

$$\begin{aligned} d_2^2 + d_1 d_2 - d_2 + \frac{1}{2}d_1^2 - 2d_1 + \frac{5}{2} &= d_1^2 + d_1 d_2 - d_1 + \frac{1}{2}d_2^2 - 2d_2 + \frac{5}{2} \\ 0 &= \frac{1}{2}d_2^2 + d_2 - \frac{1}{2}d_1^2 - d_1. \end{aligned}$$

So $d_2 = \frac{-1 \pm \sqrt{1-2(-d_1-\frac{1}{2}d_1^2)}}{1} = -1 \pm \sqrt{(d_1+1)^2} = -1 \pm (d_1+1)$. This occurs when the value of $d_1 = d_2$ or when $d_2 = -2 - d_1$. If $d_2 = -2 - d_1$, the displacement simplex will have scale $2 - d_1 - (-2 - d_1) = 4$. As this simplex will force us to have a higher multilattice covering density than our lattice covering density, it must lie outside of the region of $\mathbb{R}^2$ such that the density will be less than that of the lattice density. So these two functions agree when $d_1 = d_2$. We may write the density as the piecewise function

$$\delta = \begin{cases} d_1^2 + d_1 d_2 - d_1 + \frac{1}{2}d_2^2 - 2d_2 + \frac{5}{2} & \text{if } d_1 \geq d_2 \\ d_2^2 + d_1 d_2 - d_2 + \frac{1}{2}d_1^2 - 2d_1 + \frac{5}{2} & \text{if } d_2 > d_2. \end{cases}$$

We then set $\delta = c$, and the region which satisfies this equation will be a level curve. Call this region C. If we choose our displacement from the interior of C, the density of our multilattice covering will be less than c. We have seen that if $d_1 = d_2$, then the density is given by a quadratic function whose leading coefficient is positive. So any point on the interior of C that lies on the line $y = x$ will give a lower density than c.

Now let us consider the case when $d_1 > d_2$. Let us define the function $f(d_1, d_2) =$

$d_1^2 + d_1 d_2 - d_1 + \frac{1}{2}d_2^2 - 2d_2 + \frac{5}{2}$. As this defines our function when $d_1 \geq d_2$, we take the directional derivative in the direction of $u = (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. In other words, the directional derivative in the direction of the line $y = x$. We get

$$D_u f(d_1, d_2) = \frac{-1}{\sqrt{2}} - \frac{d_1}{\sqrt{2}}.$$

As $d_1 > 0$, this must be negative.

If $d_1 < d_2$, then we let $g(d_1, d_2) = d_2^2 + d_1 d_2 - d_2 + \frac{1}{2}d_1^2 - 2d_1 + \frac{5}{2}$. Then by a similar argument, the directional derivative of $D_{-u}g(d_1, d_2)$ must also be negative. So the interior of C must return a value less than c. $\square$

So suppose that we wanted to find where the density would be 1.7. We need only set δ equal to 1.7, and find the resulting d_1 and d_2 coordinates. Notice that if $d_1 = d_2$, then we find where

$$0 = \frac{4}{5} - 3d + \frac{5}{2}d^2.$$

Which occurs when $d = \frac{2}{5}$ or $\frac{4}{5}$. The points with $d_1 > d_2$ where the density is 1.7 are the points where

$$1.7 = d_1^2 + d_1 d_2 - d_1 + \frac{1}{2}d_2^2 - 2d_2 + \frac{5}{2}$$

or where

$$d_2 = 2 - d_1 - \sqrt{2.4 - 2d_1 - d_1^2}.$$

Similarly, when $d_2 > d_1$ we get a density of 1.7 when

$$1.7 = d_2^2 + d_1 d_2 - d_2 + \frac{1}{2}d_1^2 - 2d_1 + \frac{5}{2}$$

or when

$$d_2 = \frac{1 - d_1 + \sqrt{-2.2 + 6d_1 - d_1^2}}{2}.$$

Notice that our point $\left(\frac{3}{5}, \frac{3}{5}\right)$ is the center of this region. This is illustrated in Figure 4.1. We

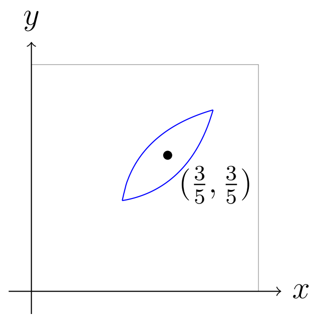

Figure 4.1: Region of density at most 1.7

note that for any chosen value of the density function between 1.6 and 2, we may follow the same process to find the bound of the region.

4.2 Rectangular tile

Once more we would like to know what the region looks like in a broader situation. Let us examine when we have a rectangular tile.

Proposition 4.2. *Let T be a rectangular tile of a lattice L such that* $\mathrm{Mdiam}(T) = 1$. *Let (x', y') be the outermost corner of our tile. Let c be some value such that $c < \delta_L$. Let $D = \{0, (x, y)\}$. Let C be the curve defined by*

$$c = \begin{cases} \frac{(5-4y')+(-4)h_1+(-2-4x')h_2+(2)h_2^2+(2)h_1h_2+h_1^2}{2x'-2x'^2} & \text{if } x'+d_2 \geq y'+d_1 \\ \frac{(1)+(-2-4x')h_1+(2)h_2+h_2^2+(2)h_1h_2+(2)h_1^2}{2x'-2x'^2} & \text{if } x'+d_2 < y'+d_1. \end{cases}$$

Then for any choice of (x, y) such that $x = x' - h_1$ and $y = y' - h_2$ for the point (h_1, h_2) that lies on the interior of C, the value of δ_{L+D} is at most c.

Proof. Let the tile have dimensions $x' \times y'$. Once more let us scale the tile and lattice until the $x' + y' = 1$. Then we note that we may once more place the displacement at the point (d_1, d_2) where $d_1 < x'$ and $d_2 < y'$. Then the the areas of the base and displacement simplices and the tile are

$$S_B = \frac{(\max\{x' + d_2, y' + d_1\})^2}{2}$$
$$S_D = \frac{(1 - d_1 - d_2)^2}{2}$$
$$A_T = x'y'.$$

Using the fact that $y' = 1 - x'$, we note that the density function becomes

$$\delta = \frac{x'^2 + 2x'd_2 + 2d_2^2 + 1 - 2d_1 - 2d_2 + d_1^2 + 2d_1d_2}{2x' - 2x'^2}$$

when $x' + d_2 \geq y' + d_1$. Otherwise the density is given by

$$\delta = \frac{2 - 2x' - 2x'd_1 + x'^2 + 2d_1^2 - 2d_2 + 2d_1d_2 + d_2^2}{2x' - 2x'^2}.$$

These two functions are equal when

$$x'^2 + 2x'd_2d_2^2 = 1 - 2x' + 2d_1 - 2x'd_1 + x'^2 + d_1^2$$

or when

$$d_2 = d_1 + 1 - 2x' \text{ or } d_2 = -d_1 - 1.$$

To see which of these we must choose, let $h_1 = x' - d_1$ and $h_2 = y' - d_2$. Then we note that when the previous holds, we also have

$$y' - h_2 = x' - h_1 + 1 - 2x'$$
$$h_1 = h_2.$$

So as before, the intersection of the two functions lie on the line $y = x - x' + y'$. So we have that

$$\delta = \begin{cases} \frac{x'^2+1}{2x'-2x'^2} + \frac{2x'-2}{2x'-2x'^2}d_2 + \frac{2}{2x'-2x'^2}d_2^2 + \frac{2}{2x'-2x'^2}d_1 + \frac{2}{2x'-2x'^2}d_1d_2 + \frac{1}{2x'-2x'^2}d_1^2 \\ \qquad \text{if } x'+d_2 \geq y'+d_1 \\ \frac{2-2x'+x'^2}{2x'-2x'^2} + \frac{-2}{2x'-2x'^2}d_2 + \frac{1}{2x'-2x'^2}d_2^2 + \frac{-2x'}{2x'-2x'^2}d_1 + \frac{2}{2x'-2x'^2}d_1d_2 + \frac{2}{2x'-2x'^2}d_1^2 \\ \qquad \text{if } x'+d_2 < y'+d_1. \end{cases}$$

Notice that if we scale our multilattice by a factor of $2x' - 2x'^2$, we get that

$$\delta = \begin{cases} x'^2 + 1 + (2x'-2)d_2 + 2d_2^2 + 2d_1 + 2d_1d_2 + d_1^2 & \text{if } x'+d_2 \geq y'+d_1 \\ x'^2 - 2x' + 2 - 2d_2 + d_2^2 - 2x'd_1 + 2d_1d_2 + 2d_1^2 & \text{if } x'+d_2 < y'+d_1. \end{cases}$$

Rather than express this in terms of d_i, let us switch again to the notation of h_i. Then the density becomes

$$\delta = \begin{cases} x'^2 + 1 + (2x'-2)(y'-h_2) + 2(y'-h_2)^2 + 2(x'-h_1) + 2(x'-h_1)(y'-h_2) \\ \qquad +(x'-h_1)^2 w \qquad \text{if } x'+d_2 \geq y'+d_1 \\ x'^2 - 2x' + 2 - 2(y'-h_2) + (y'-h_2)^2 - 2x'(x'-h_1) + 2(x'-h_1)(y'-h_2) \\ \qquad +2(x'-h_1)^2 \qquad \text{if } x'+d_2 < y'+d_1. \end{cases}$$

Which we simplify to

$$\delta = \begin{cases} (2x'^2 + 4x'y' + 2y'^2 + 2x' - 2y' + 1) + (-2 - 2y' - 2x')h_1 + (-2 - 4x')h_2 + \\ (2)h_2^2 + (2)h_1h_2 + h_1^2 \\ (x'^2 + 2x'y' + y'^2 - 2x' - 2y' + 2) + (-6x' - 2y')h_1 + (4 - 2y' - 2x')h_2 \\ h_2^2 + (2)h_1h_2 + (2)h_1^2. \end{cases}$$

We note that if we scale back to our original lattice, we may use the fact that $x' + y' = 1$ to simplify the two cases further and we get

$$\delta = \begin{cases} \frac{(5-4y')+(-4)h_1+(-2-4x')h_2+(2)h_2^2+(2)h_1h_2+h_1^2}{2x'-2x'^2} & \text{if } x'+d_2 \geq y'+d_1 \\ \frac{(1)+(-2-4x')h_1+(2)h_2+h_2^2+(2)h_1h_2+(2)h_1^2}{2x'-2x'^2} & \text{if } x'+d_2 < y'+d_1. \end{cases} \qquad \square$$

4.3 The boundary on an L-shaped tile

The other option is to look at what happens with the tile when we have an L-shaped tile, and we are trying to bound density. As before we may scale this to our needs. Let the outer corners of the tile have coordinates (a, d) and (c, b). Let $a < c$ and $b < d$. The squinch has coordinates (a, b). Without loss of generality, let $a + d \geq c + b$. Then let our displacement be $d = (d_1, d_2) = (a - h_1, d - h_2)$. We note that if we do not choose two positive numbers h_1 and h_2, then the diameter of the base simplex will be the same as the diameter for the simplex used in the lattice covering. Thus any displacement that improves density may be written in this way. We now break this into two cases for an improvement on density of our tile.

The first case is that we pull back only the outermost corner, (a, d). In this case we get the following.

Proposition 4.3. *Let L be a lattice such that the tile T obtained from the subtraction construction is an L-shaped tile. Furthermore, let* $\mathrm{Mdiam}\,(T) = 1$. *Let the tile T have the corners at (a, d) and (c, b) such that $a + d = 1$. Let the curve C be defined by the region*

$$c = \begin{cases} \frac{(c+b)^2+(h_1+h_2)^2}{2(ad+bc-ab)} & \textit{if } c+b \geq 1-h_i \\ \frac{1-2h_1+2h_1^2+2h_1h_2+h_2^2}{2ad+2cb-2ab} & \textit{if } 1-h_1 > c+b \textit{ and } h_1 < h_2 \\ \frac{1-2h_2+h_1^2+2h_1h_2+2h_2^2}{2ad+2cb-2ab} & \textit{if } 1-h_2 > c+b \textit{ and } h_2 < h_1. \end{cases}$$

for some constant c. Let (h_1, h_2) lie on the interior of C. Then for any set $D = \{0, (a - h_1, d - h_2)\}$, $\delta_{L+D} \leq c$.

Proof. As we have seen, if we are pulling back just the corner (a, d), our choice is optimized when $h_1 = h_2$. We also note that $h_1 \leq a+d-c+b$ for our optimal covering density. We note that this will still be the most efficient way to pull back a single corner to improve density. However, we may still choose $h_1 \neq h_2$ and improve our density by some amount. This results in a Manhattan diameter of the base simplex to be $\max\{a + d - h_1, a + d - h_2, c + b\}$. The displacement will have a diameter of h_1+h_2. Let us scale the multilattice such that $a+d = 1$. So the density of our multilattice covering will be

$$\delta = \frac{(\max\{1 - h_1, 1 - h_2, c + b\})^2 + (h_1 + h_2)^2}{2(ad + cb - ab)}.$$

Let us take the case where $a + d - h_2$ is the largest. Then our density function is given by

$$\delta = \frac{(1 - h_2)^2 + (h_1 + h_2)^2}{2(ad + cb - ab)} = \frac{1 - 2h_2 + h_1^2 + 2h_1h_2 + 2h_2^2}{2ad + 2cb - 2ab}.$$

Now if we look at the region where $a + d - h_1$ is the largest, we get

$$\delta = \frac{(1 - h_1)^2 + (h_1 + h_2)^2}{2(ad + cb - ab)} = \frac{1 - 2h_1 + 2h_1^2 + 2h_1h_2 + h_2^2}{2ad + 2cb - 2ab}.$$

Let us now look at how these two functions will interact. The two are equal when

$$1 - 2h_1 + 2h_1^2 + 2h_1h_2 + h_2^2 = 1 - 2h_2 + h_1^2 + 2h_1h_2 + 2h_2^2.$$

This happens when

$$-2h_1 + h_1^2 = -2h_2 + h_2^2.$$

Note that this gives the two possibilities that $h_2 = h_1$ or $h_2 = 2-h_1$. As $2-h_1 = 2(a+d)-h_1$, we need not worry about the second option. So these two agree when $h_1 = h_2$. Notice that the density is expressed in terms of h_i, so will be the distance in the negative direction from the corner (a, d). Note that this will result in a region of a similar shape to the rectangular case. However, we also note that rather than just looking at how far back in each direction the displacement is from the outermost corner, we also worry about what the diameter of the second corner is. Now we look at what happens if $c+b$ is the greatest value of the three. We now have that the base simplex will have a diameter of $c+b$. This is fixed, even if we vary the displacement slightly. So our density then becomes

$$\delta = \frac{(c+b)^2 + (h_1+h_2)^2}{2(ad+bc-ab)}.$$

Now we note that if we want δ to be less than a constant number m,

$$\begin{aligned}
&m > \frac{(c+b)^2 + (h_1+h_2)^2}{2(ad+bc-ab)} \\
&2m(ad+bc-ab) > (c+b)^2 + (h_1+h_2)^2 \\
&2m(ad+bc-ab) - (c+b)^2 > (h_1+h_2)^2 \\
&\sqrt{2m(ad+bc-ab) - (c+b)^2} > h_1+h_2.
\end{aligned}$$

We must obviously choose the positive root for the left, if it is a real number. Note that any combination of h_1 and h_2 which satisfy this will lie above a translation of the line $y = -x$. So, we note that this precludes any options that we have that would lie below the line $y = -x + 1 - \sqrt{x2m(ad+bc-ab) - (c+b)^2}$. This further restricts our region of a given density. So our density function is determined by the Manhattan distance from our displacement to the outermost corner when $c+b$ is the diamter of the base tile. We get the piecewise function

$$\delta = \begin{cases} \frac{(c+b)^2+(h_1+h_2)^2}{2(ad+bc-ab)} & \text{if } c+b \geq 1-h_i \\ \frac{1-2h_1+2h_1^2+2h_1h_2+h_2^2}{2ad+2cb-2ab} & \text{if } 1-h_1 > c+b \text{ and } h_1 < h_2 \\ \frac{1-2h_2+h_1^2+2h_1h_2+2h_2^2}{2ad+2cb-2ab} & \text{if } 1-h_2 > c+b \text{ and } h_2 < h_1. \end{cases} \qquad \square$$

To see this region, let us work out an example. We will only look at the region of the

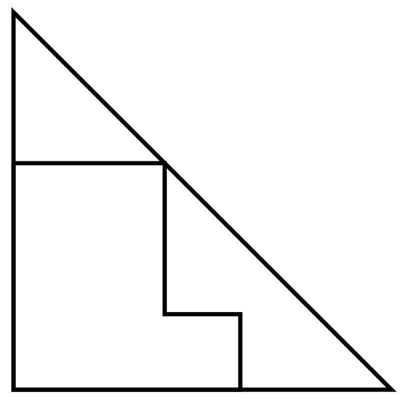

Figure 4.2: L-tile generated by (0.4,0.2) and (-0.2, 0.6)

tile that has y values greater than or equal to that of the squinch. Let us take the lattice generated by $v_1 = (0.4, 0.2)$ and $v_2 = (-0.2, 0.6)$ We note that the tile generated by this will have area 0.28 or $\frac{7}{25}$. Now we note that the outermost corner of this tile will lie at $(0.4, 0.6)$. The second corner will be at $(0.6, 0.2)$. Figure 4.2 illustrates this tile.

If we place a displacement at the point $(0.2, 0.4)$ we obtain the minimum density possible with one displacement. This density is $\frac{10}{7}$. However, instead of $\frac{10}{7}$, what if I wanted a density of $\frac{8}{5}$? The lattice covering results in a density higher than this. So we now look at the region of this tile that would result in a density of 1.6 or less. To do this we first look at when $h_1 \geq h_2$ and $1 - h_1 > 0.8$. The formula for density here is

$$\frac{1 - 2h_2 + h_1^2 + 2h_1h_2 + 2h_2^2}{0.56}.$$

Next we note that if $h_2 \geq h_1$ and $1 - h_2 > 0.8$, we get

$$\frac{1 - 2h_1 + h_2^2 + 2h_1h_2 + 2h_1^2}{0.56}.$$

Finally we note that if $0.8 \geq 1 - h_i$, we get

$$\frac{0.64 + (h_1 + h_2)^2}{0.56}.$$

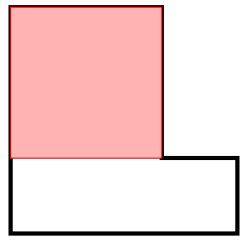

Figure 4.3: Region to pull back one corner

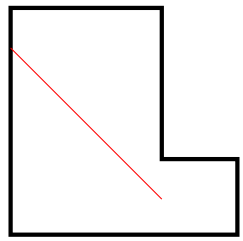

Figure 4.4: Boundary for $h_1 + h_2$

When we solve for when each of these is at most 1.6, we get the following. Notice that as we begin, we have that the possible displacements must lie in the highlighted portion of the tile in Figure 4.3.

When $0.8 \geq 1 - h_i$,

$$\begin{aligned}
1.6 &\geq \frac{0.64 + (h_1 + h_2)^2}{0.56} \\
0.896 &\geq 0.64 + (h_1 + h_2)^2 \\
0.256 &\geq (h_1 + h_2)^2 \\
\sqrt{0.256} &\geq h_1 + h_2.
\end{aligned}$$

This limits the possible values of our tile to be above the line $y = -x + 1 - \sqrt{0.256}$. This line is drawn on the tile in Figure 4.4.

Now if we look at when $h_1 \geq h_2$, and $1 - h_2 > 0.8$, the density is 1.6 when

$$\begin{aligned}
1.6 &= \frac{1 - 2h_2 + h_1^2 + 2h_1h_2 + 2h_2^2}{0.56} \\
0.896 &= 1 - 2h_2 + h_1^2 + 2h_1h_2 + 2h_2^2.
\end{aligned}$$

If we solve for h_1, we get

$$h_1 = \sqrt{-h_2^2 + 2h_2 - 0.104} - h_2.$$

When $h_2 > h_1$, the density is 1.6 when

$$1.6 = \frac{1 - 2h_1 + h_2^2 + 2h_1h_2 + 2h_1^2}{0.56}$$
$$0.896 = 1 - 2h_1 + h_2^2 + 2h_1h_2 + 2h_1^2.$$

Solving this for h_2 we get

$$h_2 = \sqrt{-h_1^2 + 2h_1 - 0.104} - h_1.$$

We note that these are symmetric about the line $y = x + 0.2$, or the line that passes through (a, d) with a slope of 1. We note that

$$\sqrt{-h_1^2 + 2h_1 - 0.104} - h_1 = -x + \sqrt{0.256}$$

when $h_1 = 0.2$. So we note that these three curves will bound the region of density. Figure 4.5 illustrates the pieces of the density function shown, together with the plane $z = 1.6$.

It may not always be the case that an decrease in covering density results from a multilattice whose displacement simplex covers only the corner (a, d). It may be the case that the displacement simplex covers both (a, d) and (c, b). If we choose y to be small enough, specifically $y < d - b$, then $y < b$, and the point (c, b) will be covered by our simplex placed at (x, y). Thus we pull back two corners with one displacement. We have already shown that this will not be the best density possible with the given number of displacements, but it still may give a covering with a lower density than that of the lattice covering.

Proposition 4.4. *Let L be a lattice such that the tile T obtained from the subtraction construction is an L-shaped tile and* $\mathrm{Mdiam}\,(T) = 1$*. Then let c be a constant. Let C be the*

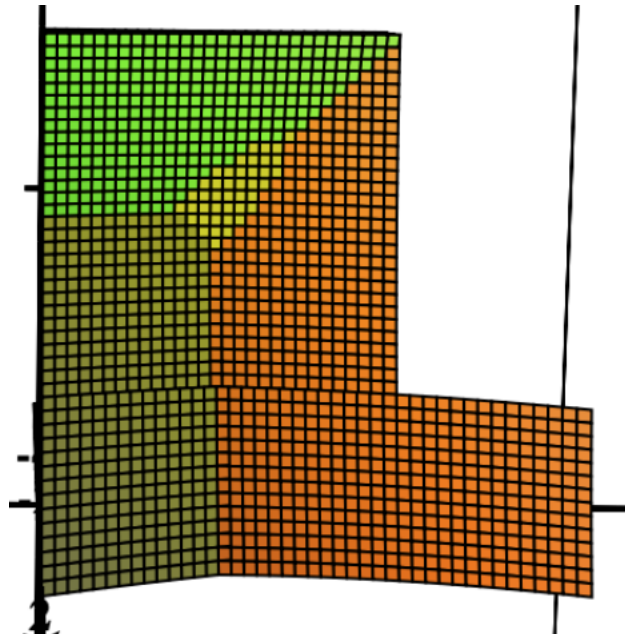

Figure 4.5: Region of density 1.6 or less

curve defined by

$$c = \begin{cases} \frac{1+d^2-2db+b^2+(-2+2d-2b)\Gamma_1+2\Gamma_1\Gamma_2+2\Gamma_1^2+(2d-2b)\Gamma_2+\Gamma_2^2}{2(ad+bc-ab)} & \text{if } 1-\Gamma_1 \geq c+b-\Gamma_2 \\ \frac{d^2-2bd+2b^2+2cb+c^2+(2d-2b)\Gamma_1+(2d-4b-2c)\Gamma_2+\Gamma_1^2+2\Gamma_1\Gamma_2+2\Gamma_2^2}{2(ad+bc-ab)} & \text{if } c+b-\Gamma_2 > 1-\Gamma_1. \end{cases}$$

Choose any Γ_1, Γ_2 *such that* (Γ_1, Γ_2) *lies on the interior of* C*. Let* $D = \{0, (a-\Gamma_1, b-\Gamma_2)\}$*. Then the multilattice covering density* $\delta_{L+D} < c$*.*

Proof. Let $a + d = 1$, so that the lattice covering will have a simplex of diameter 1. Then to pull back both corners, we must place our displacement such that it has coordinates $(a - \Gamma_1, b - \Gamma_2)$ such that $\Gamma_1 > 0$ and $\Gamma_2 > 0$. Let us first find the displacement that will give the minimum covering density and will cover (a, d) and (c, b). To minimize the density, we want to choose Γ_1 and Γ_2 in such a way that $a + d - \Gamma_1 = b + c - \Gamma_2$. This will ensure that the corners of the prototile covered by the base simplex will have the same Manhattan diameter. We also note that this will cause the highlighted region in Figure 4.7 to be where our displacement may be. We know this as $\Gamma_1 \geq \Gamma_2$. Note that the density of the multilattice

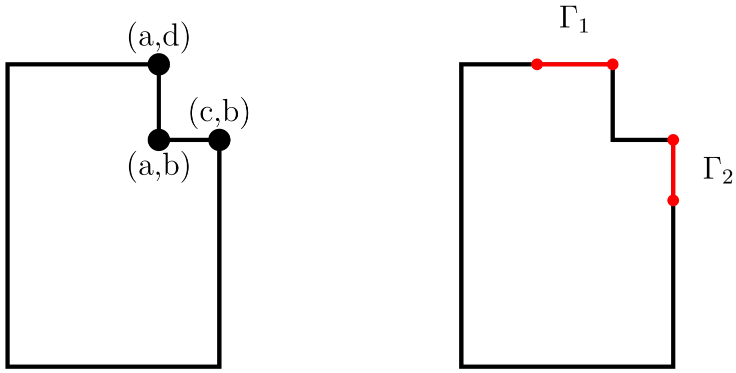

Figure 4.6: Γ_1 and Γ_2 on L-tile

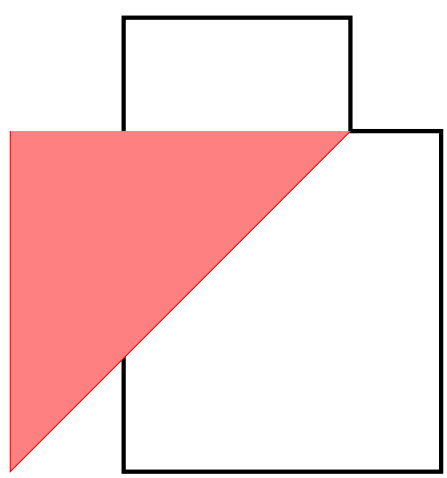

Figure 4.7: Region to pull back both corners optimally

will then become

$$\begin{aligned}
\delta &= \frac{(a+d-\Gamma_1)^2 + (\Gamma_1+\Gamma_2+d-b)^2}{2(ad+bc-ab)} \\
&= \frac{a^2+2ad+2d^2-2db+b^2+2\Gamma_1\Gamma_2+(-2a-2b)\Gamma_1+2\Gamma_1^2+(2d-2b)\Gamma_2+\Gamma_2^2}{2(ad+bc-ab)} \\
&= \frac{1+d^2-2db+b^2+(-2+2d-2b)\Gamma_1+2\Gamma_1\Gamma_2+2\Gamma_1^2+(2d-2b)\Gamma_2+\Gamma_2^2}{2(ad+bc-ab)}.
\end{aligned}$$

This is dependent on the distance from the outermost corner to the squinch, the Manhattan diameter of the tile, and the area of the tile. We also note that by our choices, $\Gamma_2 = b+c-1+\Gamma_1$. So we may express density as

$$\begin{aligned}
\delta &= \frac{(1-\Gamma_1)^2+(2\Gamma_1+c-a)^2}{2(ad+bc-ab)} \\
&= \frac{1+c^2-2ac+a^2+(4c-4a-2)\Gamma_1+5\Gamma_1^2}{2(ad+bc-ab)}.
\end{aligned}$$

Taking the derivative of the density function with respect to Γ_1 we get

$$\delta' = \frac{4c-4a-2+10\Gamma_1}{2(ad+bc-ab)}.$$

We note that $\frac{1-2c+2a}{5}$ is the only critical number we get. Now we note that for Γ_1 less than this value, δ' returns negative numbers. If Γ_1 is greater then we get positive numbers. Thus this is a local minimum. So the best density we can get by covering an L-shaped tile will be achieved when we place the displacement at $\left(\frac{3a+2c-1}{5}, \frac{4-3c-2a}{5}\right)$. We note that this point must lie behind the squinch. So we also note that $a > \frac{3a+2c-1}{5}$ and also that $b > \frac{4-3c-2a}{5}$. If this is not the case, we cannot improve the density by placing a displacement which covers an L-shaped portion of the tile.

Moving forward, let us assume that we may improve the density of our multilattice covering with a displacement that covers an L-shaped portion of our tile. In other words, a and b satisfy the previous inequalities. Now rather than achieving the best possible density, let us look at the region that will give us a multilattice covering below a certain value. We note that the Manhattan diameter of the base simplex will be $\max\{1-\Gamma_1, c+b-\Gamma_2\}$. We also note that the diameter of the displacement simplex will be $\Gamma_1+\Gamma_2+d-b$. So our density function will be

$$\delta = \frac{(\max\{1-\Gamma_1, c+b-\Gamma_2\})^2 + (\Gamma_1+\Gamma_2+d-b)^2}{2(ad+bc-ab)}.$$

Let us consider when $1-\Gamma_1 \geq c+b-\Gamma_2$. Then we have

$$\delta = \frac{1+d^2-2db+b^2+(-2+2d-2b)\Gamma_1+2\Gamma_1\Gamma_2+2\Gamma_1^2+(2d-2b)\Gamma_2+\Gamma_2^2}{2(ad+bc-ab)}.$$

Now let us consider when $c+b-\Gamma_2 > 1-\Gamma_1$. Our density function is now given by

$$\begin{aligned}\delta &= \frac{(c+b-\Gamma_2)^2+(\Gamma_1+\Gamma_2+d-b)^2}{2(ad+bc-ab)} \\ &= \frac{d^2-2bd+2b^2+2cb+c^2+(2d-2b)\Gamma_1+(2d-4b-2c)\Gamma_2+\Gamma_1^2+2\Gamma_1\Gamma_2+2\Gamma_2^2}{2(ad+bc-ab)}.\end{aligned}$$

We note that these two functions will agree when

$$1-\Gamma_1 = c+b-\Gamma_2$$

$$\Gamma_2 = c+b+\Gamma_1-1.$$

This will be the line of slope 1 that passes through the point $\left(\frac{3a+2c-1}{5}, \frac{4-3c-2a}{5}\right)$. So we once again get a shape that is symmetric about the line of slope 1 passing through the displacement that gives a local minimum covering density. $\square$

Let us work out an example of this type. In order to improve the covering density of a given tile, we need the corners to be relatively close together. Let us take the lattice generated by $v_1 = \left(\frac{6}{16}, \frac{8}{16}\right)$ and $v_2 = \left(\frac{-1}{16}, \frac{10}{16}\right)$. The covolume of the lattice will be $\frac{68}{256}$. The corners of the tile will be at $\left(\frac{6}{16}, \frac{10}{16}\right)$ and $\left(\frac{7}{16}, \frac{8}{16}\right)$. By our above calculations, the best density will result when our displacement is at $\left(\frac{1}{5}, \frac{31}{80}\right)$. This density will be about 1.60147. Now let us look at the region behind the squinch that will give us a density of 1.65. First let

$1 - \Gamma_1 > \frac{15}{16} - \Gamma_2$. The density will be

$$\begin{aligned}
\delta &= \frac{(1-\Gamma_1)^2 + \left(\frac{2}{16} + \Gamma_1 + \Gamma_2\right)^2}{2\left(\frac{68}{256}\right)} \\
&= \frac{260 - 448\Gamma_1 + 64\Gamma_2 + 512\Gamma_1^2 + 512\Gamma_1\Gamma_2 + 256\Gamma_2^2}{136}.
\end{aligned}$$

If we want this to be equal to 1.65, then we get

$$\begin{aligned}
1.65 &= \frac{260 - 448\Gamma_1 + 64\Gamma_2 + 512\Gamma_1^2 + 512\Gamma_1\Gamma_2 + 256\Gamma_2^2}{136} \\
\Gamma_2 &= \frac{8\sqrt{-\Gamma_1^2 + 2\Gamma_1 - 0.123438} - 8\Gamma_1 - 1}{8}.
\end{aligned}$$

Now if we have $\frac{15}{16} - \Gamma_2 > 1 - \Gamma_1$, the density is

$$\begin{aligned}
\delta &= \frac{\left(\frac{15}{16} - \Gamma_2\right)^2 + \left(\frac{2}{16} + \Gamma_1 + \Gamma_2\right)^2}{2\left(\frac{68}{256}\right)} \\
&= \frac{229 + 64\Gamma_1 - 416\Gamma_2 + 256\Gamma_1^2 + 512\Gamma_1\Gamma_2 + 512\Gamma_2^2}{136}.
\end{aligned}$$

To get this to be 1.65, we get

$$\begin{aligned}
1.65 &= \frac{229 + 64\Gamma_1 - 416\Gamma_2 + 256\Gamma_1^2 + 512\Gamma_1\Gamma_2 + 512\Gamma_2^2}{136} \\
\Gamma_2 &= \frac{-16\sqrt{-\Gamma_1^2 - \frac{17}{8}\Gamma_1 + 0.624} - 16\Gamma_1 + 13}{32}.
\end{aligned}$$

So we get

$$\Gamma_2 = \begin{cases} \frac{8\sqrt{-\Gamma_1^2 + 2\Gamma_1 - 0.123438} - 8\Gamma_1 - 1}{8} & \text{if } \Gamma_1 \leq \frac{1}{16} + \Gamma_2 \\ \frac{-16\sqrt{-\Gamma_1^2 - \frac{17}{8}\Gamma_1 + 0.624} - 16\Gamma_1 + 13}{32} & \text{if } \Gamma_1 > \frac{1}{16} + \Gamma_2. \end{cases}$$

The point $(a - x, b - y)$ for any (x, y) that lies on the interior of the region defined by the above will result in a multilattice covering density of at most 1.65.

While these results may be a good start for someone interested in multilattices, there is still much to be explored.

One question that is closely related but is still unexplored is that of how to choose optimal generators given a certain number of displacements. For example, if I take 5 displacements, what two generators should I choose for a group of size k? We have shown that the most efficient generators of the group will not in general be the best generators to use to obtain a multilattice covering of lowest density.

Another question that would require a little more time and resources is that of determining the best covering in $n = 2$ for a low number of displacements. This problem may be approached using multivariate calculus, but quickly grows in complexity. One approach we may use is to cover a square region as efficiently as possible, then use this to bound the density for any tile type.

It would also be interesting to explore the multilattice covering where instead of only taking dilations of the right regular simplex, we allowed for different simplices. While it is obvious that there is a choice for getting to a density of 1, it is not as obvious how rotations of a right regular simplex may form a packing or a covering. With regard to the group problem, this would be to use a different set of generators with each displacement. What would happen if we were to choose elements that did not generate the whole group in our displacement sums? Is there a bound above 1 that may be set in this instance?

Additionally, the lattice covering problem has an equivalent problem in groups. We can extend this problem to the multilattice case as well. It may be easier in this context to determine the density based on the problem in finite groups.

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